第4章 Radiative Transfer Plane-Parallel Frequency-Dependent variables $I_\nu J_\nu H_\nu K_\nu$ in astronomy

After Fukue, J. 2011, PASJ, 63, in press

4.1 Basic Equations

We here assume the followings: (i) The disk is steady and axisymmetric. (ii) It is also geometrically thin and plane-parallel. (iii) As a closure relation, we adopt the Eddington approximation. (iv) The viscous heating rate is concentrated at the equator or uniform in the vertical direction. (v) The disk atmosphere is under the local thermodynamic equilibrium (LTE). In contrast to the previous study (Fukue, Akizuki 2006), we do not assume the gray atmosphere, but treat the monocromatic radiation and scattering effect. We briefly mention the validity of the plane-parallel assumption. In the accretion disk the physical quantities have gradients in the horizontal (radial) direction. However, the ratio of the radial gradient of the physical quantities to the vertical gradient is generally on the order of z/r. Hence, the plane-parallel assumption is valid, as long as the disk is geometrically thin. This is the case for the limb-darkening except fo very small μ. In other words, the plane-parallel assumption will violate in the geometrically thick disk, such as a slim disk.

The radiative transfer equations are given in several literatures (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 2008). For the plane-parallel geometry in the vertical direction (z), the frequency-dependent transfer equation, the zeroth moment equation, and the first moment equation become, respectively,

$$\mu \frac{dI_\nu}{dz} = \rho \left[\frac{j_\nu}{4\pi} - (\kappa_\nu + \sigma_\nu) I_\nu + \sigma_\nu J_\nu \right], \tag{4.1}$$

$$\frac{dH_\nu}{dz} = \rho \left(\frac{j_\nu}{4\pi} - \kappa_\nu J_\nu \right), \tag{4.2}$$

$$\frac{dK_\nu}{dz} = -\rho(\kappa_\nu + \sigma_\nu) H_\nu, \tag{4.3}$$

where μ is the direction cosine ($= \cos\theta$), I_ν the specific intensity, J_ν the mean intensity ($= cE_\nu/4\pi$, E_ν being the radiation energy density), H_ν the Eddington flux ($H_\nu = F_\nu/4\pi$, F_ν the vertical component of the radiative flux), K_ν the mean radiation stress ($K_\nu = cP_\nu/4\pi$, P_ν the zz-component of the radiation stress tensor), ρ the gas density, and c the speed of light. The mass emissivity j_ν, the absorption opacity κ_ν, and the scattering one σ_ν generally depend on the frequency. The Eddington approximation is $K_\nu = J_\nu/3$.

Introducing the optical depth, defined by $d\tau_\nu \equiv -\rho\,(\kappa_\nu + \sigma_\nu)\,dz$, we rewrite the radiative transfer equations in the familiar form:

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu, \tag{4.4}$$

$$\frac{dH_\nu}{d\tau_\nu} = J_\nu - S_\nu, \tag{4.5}$$

$$\frac{dK_\nu}{d\tau_\nu} = \frac{1}{3}\frac{dJ_\nu}{d\tau_\nu} = H_\nu. \tag{4.6}$$

Here S_ν is the source function,

$$S_\nu = \frac{1}{\kappa_\nu + \sigma_\nu}\frac{j_\nu}{4\pi} + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu = \varepsilon_\nu B_\nu + (1 - \varepsilon_\nu) J_\nu, \tag{4.7}$$

where we assume LTE and ε_ν is the photon destruction probability:

$$\varepsilon_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}. \tag{4.8}$$

Eliminating H_ν from equations (4.5) and (4.6), we have the second order form of the transport equation:

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = J_\nu - S_\nu, = \varepsilon_\nu (J_\nu - B_\nu). \tag{4.9}$$

The energy equation for matter,

$$0 = q_{\rm vis}^{+} - \rho \int (j_\nu - 4\pi\kappa_\nu J_\nu)\, d\nu, \tag{4.10}$$

where $q_{\rm vis}^{+}$ the viscous-heating rate, is expressed as

$$\frac{q_{\rm vis}^{+}}{4\pi\rho} = \int \kappa_\nu (B_\nu - J_\nu)\, d\nu = \int (\kappa_\nu + \sigma_\nu)(S_\nu - J_\nu) d\nu. \tag{4.11}$$

Finally, the disk total optical depth becomes

$$\tau_{\nu 0} = -\int_H^0 \rho(\kappa_\nu + \sigma_\nu) dz, \tag{4.12}$$

where H is the disk half-thickness.

It should be noted that several analytical expressions for radiative moments as well as temperature distributions were obtained by several researchers under the gray approximation (e.g., Laor, Netzer 1989; Hubeny 1990; Hubeny et al. 2005; Artemova et al. 1996; Fukue, Akizuki 2006).

For the internal heating, we consider two extreme cases: (i) The viscous heating is concentrated at the disk equator and the disk atmosphere is described by a simple model. (ii) There exists uniform heating in the sense that the heating per mass is distributed in any frequency range; $j_\nu - 4\pi\kappa_\nu J_\nu$ does not depend on the optical depth.

4.2 Simple Case

We show here the simple analytical solution, which is well-known in the stellar atmosphere with semi-inifinite optical depth, in order to clarify the scattering effect and the effect of the finite optical depth in the present case.

Similar to the stellar atmosphere, we assume that the Planck function $B_\nu(\tau_\nu)$ linearly depends on the optical depth:

$$B_\nu = B_\nu(0) + b_\nu \tau_\nu, \tag{4.13}$$

and that the photon destruction probability ε_ν has the same value at all depths.

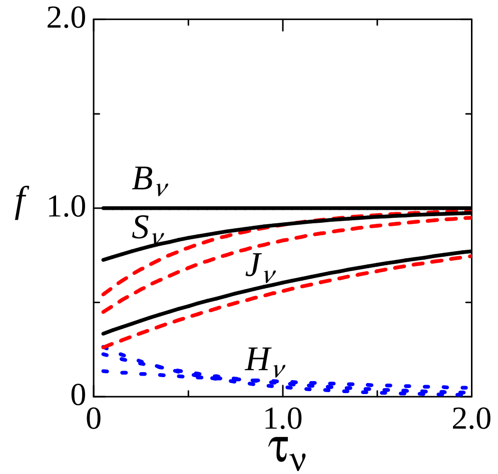

図 4.1: Radiative quantities normalized by $B_\nu(0)$ as a function of the optical depth for an isothermal case ($b_\nu = 0$). Thick dashed curves represent J_ν, thick dotted ones H_ν, and thick solid ones S_ν. The values of the parameter ε_ν are 1, 0.5, and 0.1 from top to bottom for each case.

In this case equation (4.9) can be expressed as

$$\frac{1}{3}\frac{d^2}{d\tau_\nu^2}(J_\nu - B_\nu) = \varepsilon_\nu (J_\nu - B_\nu), \tag{4.14}$$

and solved as a linear differential equation.

Imposing the boundary condition that $J_\nu = B_\nu$ for $\tau_\nu \to \infty$, and that

$$J_\nu(0) = c_\nu H_\nu(0) \text{ at } \tau_\nu = 0, \tag{4.15}$$

where $c_\nu = \sqrt{3}$, we have the following solutions (Mihalas, Mihalas 1984).

$$J_\nu = B_\nu(0) + b_\nu \tau_\nu - \frac{B_\nu(0) - b_\nu c_\nu/3}{1 + (c_\nu/3)\sqrt{3\varepsilon_\nu}} e^{-\sqrt{3\varepsilon_\nu}\tau_\nu}, \tag{4.16}$$

$$H_\nu = \frac{1}{3} b_\nu + \sqrt{\frac{\varepsilon_\nu}{3}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1 + (c_\nu/3)\sqrt{3\varepsilon_\nu}} e^{-\sqrt{3\varepsilon_\nu}\tau_\nu}, \tag{4.17}$$

$$S_\nu = B_\nu(0) + b_\nu \tau_\nu - (1 - \varepsilon_\nu) \frac{B_\nu(0) - b_\nu c_\nu/3}{1 + (c_\nu/3)\sqrt{3\varepsilon_\nu}} e^{-\sqrt{3\varepsilon_\nu}\tau_\nu}. \tag{4.18}$$

These analytical solutions normalized by the surface value $B_\nu(0)$ are shown in figures 1 and 2 for the isothermal case ($b_\nu = 0$) and the case with temperature gradient ($b_\nu/B_\nu(0) = 3/2$), respectively. The parameter c_ν is set to be $\sqrt{3}$.

In figure 1 radiative quantities for an isothermal case, which are well-known solutions, are shown. As the photon destruction probability ε_ν decreases, the source function S_ν and its surface value also decrease; this is just the scattering effect or so-called $\sqrt{\varepsilon_\nu}$-law.

In figure 2 radiative quantities for a case with temperature gradient are shown. In this case the source function and its surface value do not depend on the value of ε_ν so much, and the scattering effect is not so significant. This is also well-known in the stellar atmosphere, but has not been sufficiently recognized in the field of the accretion disk atmosphere, as already mentioned in the introduction.

Using above solutions, we can also solve the transfer equation (4.4) to obtain the intensity $I_\nu(\tau_\nu)$. In a geometrically thin disk with finite optical depth $\tau_{\nu 0}$ and uniform incident intensity $I_{\nu 0}$ from the disk equator, the boundary value

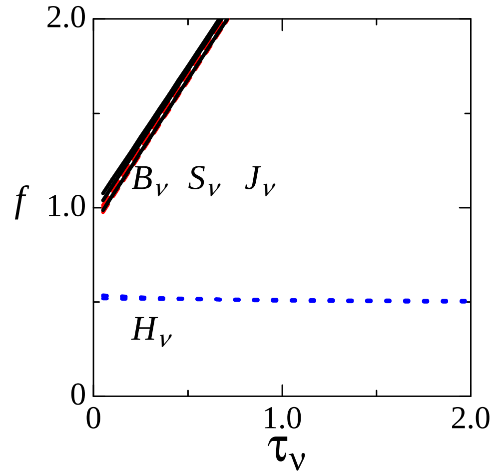

図 4.2: Radiative quantities normalized by $B_\nu(0)$ as a function of the optical depth for a case with temperature gradient ($b_\nu/B_\nu(0) = 3/2$). Thick dashed curves represent J_ν, thick dotted ones H_ν, and thick solid ones S_ν. Thin solid curves denote the Planck function B_ν. The values of the parameter ε_ν are 1, 0.5, and 0.1 from top to bottom for each case.

$I_\nu(\tau_{\nu 0}, \mu)$ of the outward intensity consists of two parts:

$$I_\nu(\tau_{\nu 0}, \mu) = I_{\nu 0} + I_\nu(\tau_{\nu 0}, -\mu), \tag{4.19}$$

where $I_{\nu 0}$ is the uniform incident intensity and $I_\nu(\tau_{\nu 0}, -\mu)$ is the *inward* intensity from the backside of the disk beyond the midplane (Fukue, Akizuki 2006).

After some manipulations, we finally obtain the outward intensity for the present simple case with finite optical depth.

$$\begin{aligned} I_\nu(\tau_\nu, \mu) &= I_{\nu 0} e^{(\tau_\nu - \tau_{\nu 0})/\mu} + B_\nu(0) + b_\nu \mu + b_\nu \tau_\nu \\ & \quad -2b_\nu \mu e^{(\tau_\nu - \tau_{\nu 0})/\mu} - [B_\nu(0) - b_\nu \mu] e^{(\tau_\nu - 2\tau_{\nu 0})/\mu} \\ & -\frac{1-\varepsilon_\nu}{1+\mu\sqrt{3\varepsilon_\nu}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}} \left[e^{-\sqrt{3\varepsilon_\nu}\tau_\nu} - e^{\tau_\nu/\mu - (1+\mu\sqrt{3\varepsilon_\nu})\tau_{\nu 0}/\mu} \right] \\ & -\frac{1-\varepsilon_\nu}{1-\mu\sqrt{3\varepsilon_\nu}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}} \left[e^{\tau_\nu/\mu - (1+\mu\sqrt{3\varepsilon_\nu})\tau_{\nu 0}/\mu} - e^{(\tau_\nu - 2\tau_{\nu 0})/\mu} \right] \end{aligned} \tag{4.20}$$

Finally, the emergent intensity $I_\nu(0, \mu)$ emitted from the disk surface for the finite optical depth becomes

$$\begin{aligned} I_\nu(0, \mu) &= I_{\nu 0} e^{-\tau_{\nu 0}/\mu} + B_\nu(0) + b_\nu \mu \\ & \quad -2b_\nu \mu e^{-\tau_{\nu 0}/\mu} - [B_\nu(0) - b_\nu \mu] e^{-2\tau_{\nu 0}/\mu} \\ & -\frac{1-\varepsilon_\nu}{1+\mu\sqrt{3\varepsilon_\nu}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}} \left[1 - e^{-(1+\mu\sqrt{3\varepsilon_\nu})\tau_{\nu 0}/\mu} \right] \\ & -\frac{1-\varepsilon_\nu}{1-\mu\sqrt{3\varepsilon_\nu}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}} \left[e^{-(1+\mu\sqrt{3\varepsilon_\nu})\tau_{\nu 0}/\mu} - e^{-2\tau_{\nu 0}/\mu} \right]. \end{aligned} \tag{4.21}$$

The terms on the right-hand side of this emergent intensity (4.21) have the following meanings. The first term is the extincted intensity from the equator, the second and third terms the thermal radiation from the surface under the temperature gradient, the fourth and fifth the thermal radiation from this and backside atmospheres, respectively. The sixth term is the scattering radiation from the upper atmosphere, while the seventh is the one from the backside beyond the equator.

For sufficiently large optical depth ($\tau_{\nu 0} \to \infty$), this equation (4.21) reduces

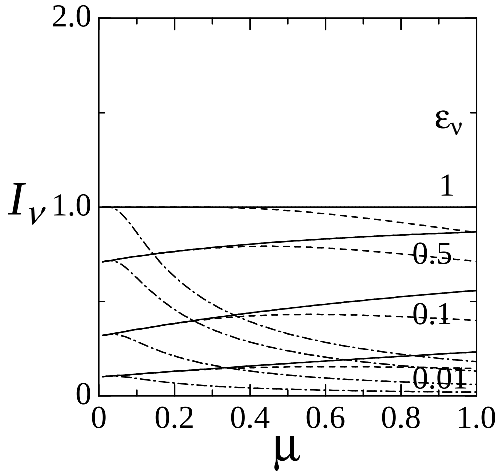

図 4.3: Normalized emergent intensity as a function of μ for an isothermal case ($b_\nu = 0$). The values of the disk optical depth $\tau_{\nu 0}$ are 10 (solid curves), 1 (dashed ones), and 0.1 (chain-dotted ones). The values of the parameter ε_ν are 1, 0.5, 0.1, and 0.01 from top to bottom for each case.

to the usual solution,

$$I_\nu(0, \mu) \sim B_\nu(0) + b_\nu \mu - \frac{1-\varepsilon_\nu}{1+\mu\sqrt{3\varepsilon_\nu}} \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}}. \tag{4.22}$$

In the limit of the small optical depth ($\tau_{\nu 0} \sim 0$), this solution (4.21) becomes as

$$I_\nu(0, \mu) \sim I_{\nu 0} \left(1 - \frac{\tau_{\nu 0}}{\mu} \right) + \left[B_\nu(0) - (1-\varepsilon_\nu) \frac{B_\nu(0) - b_\nu c_\nu/3}{1+(c_\nu/3)\sqrt{3\varepsilon_\nu}} \right] \frac{2\tau_{\nu 0}}{\mu}. \tag{4.23}$$

The first term on the right-hand side is again the uniform radiation from the equator, while the second term means the thermal and scattering radiation from the total disk thickness of both side ($2\tau_{\nu 0}$) under the limb-darkening.

The emergent intensities normalized by the surface value $B_\nu(0)$ are shown in figures 3 and 4 for the isothermal case ($b_\nu = 0$) and for the case with a

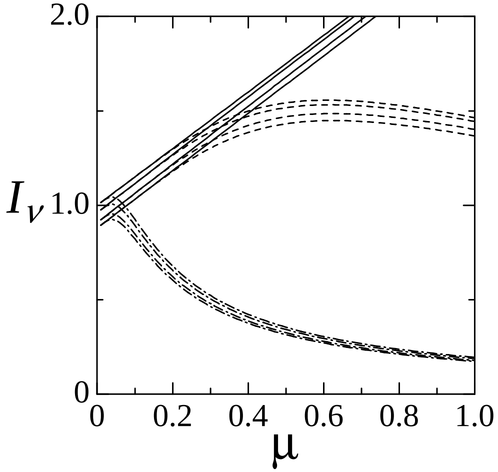

図 4.4: Normalized emergent intensity as a function of μ for a case with temperature gradient ($b_\nu/B_\nu(0) = 3/2$). The values of the disk optical depth $\tau_{\nu 0}$ are 10 (solid curves), 1 (dashed ones), and 0.1 (chain-dotted ones). The values of the parameter ε_ν are 1, 0.5, 0.1, and 0.01 from top to bottom for each case.

temperature gradient ($b_\nu/B_\nu(0) = 3/2$), respectively. The parameter c_ν is set to be $\sqrt{3}$, and the equatorial heating rate $I_{\nu 0}$ is set to be zero.

In figure 3 the emergent intensity for the isothermal case is shown. When the optical depth is sufficiently large ($\tau_{\nu 0} = 10$, solid curves), the emergent intensity decreases, as the photon destruction probability ε_ν decreases. This is again the scattering effect (the $\sqrt{\varepsilon_\nu}$-law). Furthermore, in the present case of finite optical depth, the emergent intensity towards the polar direction reduces as the disk optical depth decreases. That is, the disk becomes transparent in the vertical direction, while it is not in the edge on direction.

In figure 4 the emergent intensity for a case with temperature gradient are shown. In this case the emergent intensity does not depend on the value of ε_ν so much, and the scattering effect is not so significant. On the other hand, the effect of the finite optical depth becomes significant for small $\tau_{\nu 0}$. That is, for small optical depth the angle-dependence is drastically changed. This is because we cannot see the 'deeper' position in the atmosphere, compared with the case of a semi-infinite disk. This is a characteristic nature of the accretion disk atmosphere (cf. Hubeny 1990; Fukue, Akizuki 2006).

4.3 Uniform Heating Case

Now, we consider the case with uniform heating. In order for the right-hand of equation (4.11) to be independent of the optical depth, the necessary condition is that the term $(\kappa_\nu + \sigma_\nu)(S_\nu - J_\nu)$ does not depend on the optical depth. In the scattering dominated atmosphere, we then assume that

$$S_\nu - J_\nu = q_\nu \tag{4.24}$$

does not depend on the optical depth.

In this case we can easily integrate equations (4.5) and (4.6), under the boundary condition of $H_\nu = 0$ at $\tau_\nu = \tau_{\nu 0}$, to obtain the solutions,

$$H_\nu = H_\nu(0)\left(1 - \frac{\tau_\nu}{\tau_{\nu 0}}\right), \tag{4.25}$$

$$3K_\nu = J_\nu = H_\nu(0)\left(c_\nu + 3\tau_\nu - \frac{3\tau_\nu^2}{2\tau_{\nu 0}}\right), \tag{4.26}$$

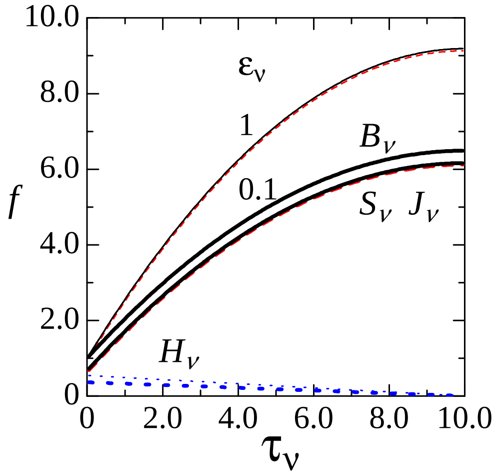

図 4.5: Radiative quantities normalized by $B_\nu(0)$ as a function of the optical depth for large optical depth ($\tau_{\nu 0} = 10$). Dashed curves represent J_ν, dotted ones H_ν, and solid ones B_ν and S_ν. The values of the parameter ε_ν are 1 (thin curves) and 0.1 (thick ones).

where $H_\nu(0) = q_\nu \tau_{\nu 0}$. These solutions for finite optical depth are essentially same as those obtained under the gray approximation in the previous studies (e.g., Laor, Netzer 1989; Hubeny 1990; Wang et al. 1999; Fukue, Akizuki 2006). It should be noted that these solutions reduce to the Milne-Eddington solution for sufficiently large optical depth.

Furthermore, the source and Planck functions become, respectively,

$$S_\nu = H_\nu(0)\left(c_\nu + 3\tau_\nu - \frac{3\tau_\nu^2}{2\tau_{\nu 0}} + \frac{1}{\tau_{\nu 0}}\right), \tag{4.27}$$

$$B_\nu = H_\nu(0)\left(c_\nu + 3\tau_\nu - \frac{3\tau_\nu^2}{2\tau_{\nu 0}} + \frac{1}{\varepsilon_\nu \tau_{\nu 0}}\right). \tag{4.28}$$

These analytical solutions normalized by the surface value $B_\nu(0)$ are shown in figures 5 and 6 for large and small optical depths, respectively. The parameter c_ν is set to be $\sqrt{3}$.

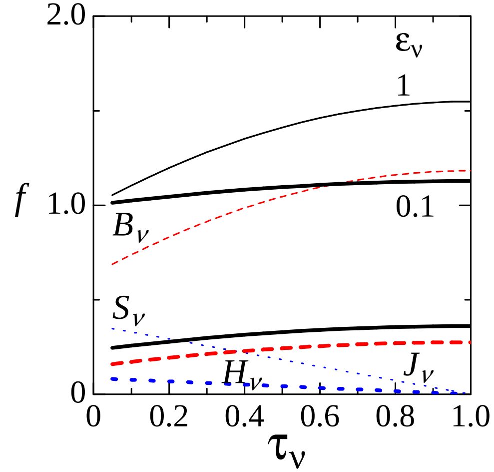

図 4.6: Radiative quantities normalized by $B_\nu(0)$ as a function of the optical depth for large small depth ($\tau_{\nu 0} = 1$). Dashed curves represent J_ν, dotted ones H_ν, and solid ones B_ν and S_ν. The values of the parameter ε_ν are 1 (thin curves) and 0.1 (thick ones).

In figure 5 radiative quantities for large optical depth ($\tau_{\nu 0} = 10$) are shown. When the photon destruction probability ε_ν is unity (thin curves in figure 5), both the mean intensity J_ν and source function S_ν are almost same as the Planck function B_ν. Indeed, in the limit of $\tau_{\nu 0} \to \infty$, $J_\nu = S_\nu = B_\nu$. When ε_ν becomes small (thick curves in figure 5), on the other hand, the mean intensity J_ν and source function S_ν are somewhat smaller than the Planck function B_ν, but the difference is not so large; the scattering effect is not significant. This is because there is a temperature gradient in the case of internal heating. In contrast to the isothermal atmosphere, where the scattering effect becomes significant, it is not so significant in the disk with internal heating and the surface temperature is close to the effective one, since the observed photons come from deep inside at the large optical depth (cf. Laor, Netzer 1989; Hui et al. 2005).

In figure 6 radiative quantities for small optical depth ($\tau_{\nu 0} = 1$) are shown. In this case the mean intensity J_ν and source function S_ν are much smaller than the Planck function B_ν. This is understood as a combination effect of the scattering and the finite optical depth. Even if there is no scattering (thin curves in figure 6), the mean intensity J_ν is smaller than the Planck function B_ν, although $S_\nu = B_\nu$. This is because of the finite optical depth; the atmosphere becomes translucent as $\tau_{\nu 0}$ decreases. Furthermore, when ε_ν becomes small, the source function S_ν as well as the mean intensity J_ν become much smaller than the Planck function B_ν. This is due to the scattering effect; in the case of finite optical depth there is no deep inside from which photons originate, and the scattering effect recovers. In such a case, similar to the isothermal atmosphere, where the $\sqrt{\varepsilon_\nu}$-law holds, the surface temperature can be remarkably larger than the effective temperature.

It should be noted that in the usual case of the isothermal atmosphere $S_\nu(0) = \sqrt{\varepsilon_\nu} B_\nu(0)$; this is the $\sqrt{\varepsilon_\nu}$-law. In the present case of finite optical depth, on the other hand,

$$S_\nu(0) = \frac{1 + c_\nu \tau_{\nu 0}}{1 + c_\nu \varepsilon_\nu \tau_{\nu 0}} \varepsilon_\nu B_\nu(0), \tag{4.29}$$

and this is *an ε_ν-law.*

As seen in analytical solutions (4.25)-(4.28), the photon destruction probability ε_ν appears only in the Planck function B_ν. Since the source function (4.27) does not include ε_ν, the solution of the intensity also does not contain ε_ν. As a result, the intensity is apparently free from the scattering effect. However, since the ratio of the coefficient of the intensity to the thermal component is

$$H_\nu(0) = \frac{\varepsilon_\nu \tau_{\nu 0}}{1 + c_\nu \varepsilon_\nu \tau_{\nu 0}} B_\nu(0), \tag{4.30}$$

there implicitly exists the scattering effect; the ε_ν-law for small optical depth.

Using above solutions, we can also solve the transfer equation (4.4) to obtain the intensity $I_\nu(\tau_\nu)$. In the case of internal heating, we impose the boundary condition at the disk midplane:

$$I_\nu(\tau_{\nu 0}, \mu) = I_\nu(\tau_{\nu 0}, -\mu). \tag{4.31}$$

After some manipulations, we finally obtain the outward intensity for the present simple case with finite optical depth (cf. Fukue, Akizuki 2006).

$$\begin{aligned} I(\tau_\nu, \mu) \quad = \quad & H_\nu(0) \Bigg[c_\nu + 3\mu + 3\tau_\nu \\ & + \frac{1}{\tau_{\nu 0}} \left(1 - 3\mu^2 - 3\mu\tau_\nu - \frac{3}{2}\tau_\nu^2 \right) \\ & - \left(c_\nu - 3\mu + \frac{1}{\tau_{\nu 0}} - \frac{3\mu^2}{\tau_{\nu 0}} \right) e^{(\tau_\nu - 2\tau_{\nu 0})/\mu} \Bigg]. \end{aligned} \tag{4.32}$$

For sufficiently large optical depth $\tau_{\nu 0}$, this equation (4.32) also reduces to the usual Milne-Eddington solution.

Finally, the emergent intensity $I_\nu(0, \mu)$ emitted from the disk surface for the finite optical depth becomes

$$\begin{aligned} I_\nu(0, \mu) \quad = \quad & H_\nu(0) \Bigg[c_\nu + 3\mu + \frac{1}{\tau_{\nu 0}} \left(1 - 3\mu^2 \right) \\ & - \left(c_\nu - 3\mu + \frac{1}{\tau_{\nu 0}} - \frac{3\mu^2}{\tau_{\nu 0}} \right) e^{-2\tau_{\nu 0}/\mu} \Bigg]. \end{aligned} \tag{4.33}$$

In the limit of the small optical depth ($\tau_{\nu 0} \sim 0$), this solution (4.33) becomes as

$$I_\nu(0, \mu) \quad \sim \quad H_\nu(0) \frac{2}{\mu} = \frac{\varepsilon_\nu B_\nu(0)}{1 + c_\nu \varepsilon_\nu \tau_{\nu 0}} \frac{2\tau_{\nu 0}}{\mu}. \tag{4.34}$$

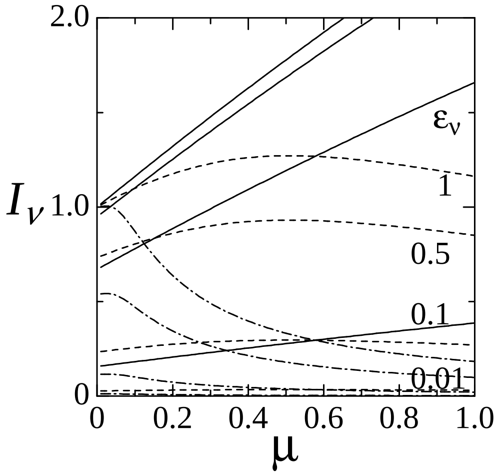

図 4.7: Normalized emergent intensity as a function of μ. The values of the disk optical depth $\tau_{\nu 0}$ are 10 (solid curves), 1 (dashed ones), and 0.1 (chain-dotted ones). The values of the parameter ε_ν are 1, 0.5, 0.1, and 0.01 from top to bottom for each case.

This is just the thermal emission from the optically thin disk with optical depth $2\tau_{\nu 0}$.

The present solutions are formally same as those of the previous result obtained by Fukue and Akizuki (2006) under the gray approximation, although the present case is frequency-dependent.

The emergent intensity normalized by the surface value $B_\nu(0)$ is shown in figures 7. The parameter c_ν is set to be $\sqrt{3}$.

As is seen in figure 7, for large optical depth ($\tau_{\nu 0} = 10$) the angle-dependence of the emergent intensity is close to the usual plane-parallel case with infinite optical depth. Therefore, the usual limb-darkening effect is seen. For small optical depth, however, the angle-dependence is drastically changed similar to the case without heating. In the vertical direction of $\mu \sim 1$, the emergent intensity decreases as the optical depth decreases. This is due to the finiteness of the optical depth. That is, we cannot see the 'deeper' position in the atmosphere, compared with the case of a semi-infinite disk. In the inclined direction of small μ, on the other hand, the emergent intensity becomes larger than that in the case of the infinite optical depth. Moreover, when the optical depth is less than unity, the emergent intensity for small μ is greater than that for large μ: the *limb brightening* takes place. In addition, as ε_ν decreases, the emergent intensity also decreases due to the scattering effect.