第11章 Relativistic Radiative Transfer Equations General : Fixed Frame variables $IEFP$ in physics

After Kato, S. et al. 2008, "Black-Hole Accretion Disks"

In this appendix we derive the basic equations for radiation hydrodynamics (photohydrodynamics) within the framework of special relativity. We first give the metric and quantities of the radiation fields, and then show the basic equations, including matter coupling.

11.1 Metric and Energy-Momentum Tensor

The full set of basic equations for photohydrodynamics can be found in several literature (e.g., Lindquist 1966; Anderson and Spiegel 1972; Hsieh and Spiegel 1976; Thorne 1981; Fukue et al. 1985; Park 2006; Takahashi 2007). It is usually expressed in a general form. In this appendix we derive and write explicitly the basic equations for relativistic radiation hydrodynamics, which are correct within the framework of special relativity. The derivation is based on Hsieh and Spiegel (1976), while correcting minor errors in their paper. In this book the $(+,-,-,-)$ signature is adopted, and the Greek suffixes α, β, γ, $\cdots$ take values of 0, 1, 2, and 3, while the Latin suffixes i, j, k, $\cdots$ take values of 1, 2, and 3. The semicolon denotes not covariant differentiation but partial differentiation, since we do not consider the space-time curvature here.

(a) **Metric**

The square of the invariant line element, ds^2, is written as

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{11.1}$$

where c is the speed of light, τ the proper time, x^μ the space-time coordinates ($x^0 = ct$ in this appendix), and $g_{\mu\nu}$ the space-time metric. The three-dimensional part of the metric, γ_{ij}, is defined by $\gamma_{ij} = -g_{ij}$.

In the case of cylindrical coordinates (r, φ, z), the line element (3.1) becomes, in a flat space-time,

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\varphi^2 - dz^2. \tag{11.2}$$

(b) **Four-velocity of matter**

The four-velocity u^μ of matter is defined by

$$u^\mu \equiv \frac{dx^\mu}{ds} = \left(\gamma, \gamma\frac{v^i}{c}\right) = \gamma\left(1, \frac{\boldsymbol{v}}{c}\right), \tag{11.3}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \tag{11.4}$$

$$v^2 = v_i v^i = \gamma_{ik} v^i v^k = -g_{ik} v^i v^k. \tag{11.5}$$

The covariant components become

$$u_\mu = g_{\mu\nu} u^\nu = \left(\gamma, -\gamma\frac{v_i}{c}\right). \tag{11.6}$$

It should be noted that $u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = 1$.

(c) **Four-momentum of photon**

The four-momentum k^μ of a photon is defined by

$$k^\mu = \left(\nu, \nu l^k\right) = \nu\left(1, \boldsymbol{l}\right), \tag{11.7}$$

where ν is the photon frequency and $\boldsymbol{l}$ is the direction cosine vector of a photon. The covariant components become

$$k_\mu = \nu\left(1, -\boldsymbol{l}\right). \tag{11.8}$$

Since $\boldsymbol{l}^2 = 1$, the contraction of the four-momentum is null:

$$k_\mu k^\mu = \nu^2 \left(1 - \boldsymbol{l}^2\right) = 0. \tag{11.9}$$

(d) Doppler and aberration effects

The four-velocity u_μ and the four-momentum k^μ expressed in comoving frames are, respectively,

$$\begin{aligned} u_\mu &= (1, 0)\,, & (11.10)\\ k^\mu &= \nu_0 \left(1, \boldsymbol{l}_0\right), & (11.11) \end{aligned}$$

where the subscript 0 means the values measured in the comoving (fluid) frame. Using equations (3.6) and (3.7), we have

$$u_\mu k^\mu = \gamma\nu - \gamma\nu \frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c} = \nu_0. \tag{11.12}$$

Thus, the transformation of the photon frequency between the inertial and comoving frames (relativistic Doppler effect) becomes

$$\begin{aligned} \nu_0 &= \nu\gamma\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right), & (11.13)\\ \nu &= \nu_0\gamma\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{l}_0}{c}\right). & (11.14) \end{aligned}$$

Similarly, the transformation of the direction cosine (relativistic aberration effect) becomes

$$\begin{aligned} \boldsymbol{l}_0 &= \frac{\nu}{\nu_0}\left[\boldsymbol{l} + \left(\frac{\gamma-1}{v^2/c^2}\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c} - \gamma\right)\frac{\boldsymbol{v}}{c}\right], & (11.15)\\ \boldsymbol{l} &= \frac{\nu_0}{\nu}\left[\boldsymbol{l}_0 + \left(\frac{\gamma-1}{v^2/c^2}\frac{\boldsymbol{v}\cdot\boldsymbol{l}_0}{c} + \gamma\right)\frac{\boldsymbol{v}}{c}\right]. & (11.16) \end{aligned}$$

The transformation of the solid angle is

$$\begin{aligned} d\Omega_0 &= \frac{\nu}{\nu_0}\frac{d\nu}{d\nu_0}d\Omega = \left[\gamma\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)\right]^{-2} d\Omega, & (11.17)\\ d\Omega &= \left[\gamma\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{l}_0}{c}\right)\right]^{-2} d\Omega_0. & (11.18) \end{aligned}$$

(e) Quantities of radiation fields

The specific intensity I_ν is related to the photon occupation number n_ν by $I_\nu = (2h\nu^3/c^2)n_\nu$. The relativistic invariant is not I_ν, but I_ν/ν^3:

$$\frac{I_\nu}{\nu^3} = \frac{I_{\nu 0}}{\nu_0^3} \equiv f. \tag{11.19}$$

Using these quantities, the energy-momentum tensor of the radiation field is defined as

$$R^{\mu\nu} \equiv \frac{2h}{c^3}\int n_\nu l^\mu l^\nu \nu^3 d\nu d\Omega = \frac{1}{c}\int I_\nu l^\mu l^\nu d\nu d\Omega, \tag{11.20}$$

where $l^\mu = (1, l^k)$. Hence, the components of $R^{\mu\nu}$ become

$$\begin{aligned} R^{00} &= \frac{1}{c}\int I_\nu d\nu d\Omega \equiv E, & (11.21)\\ R^{0i} &= \frac{1}{c}\int I_\nu l^i d\nu d\Omega \equiv \frac{1}{c}F^i, & (11.22)\\ R^{ij} &= \frac{1}{c}\int I_\nu l^i l^j d\nu d\Omega \equiv P^{ij}, & (11.23) \end{aligned}$$

where E is the radiation energy density, F^i the radiative flux, and P^{ij} the radiation stress tensor.

Integrating over the frequency, we obtain the following frequency-integrated quantities:

$$I \equiv \int I_\nu d\nu, \quad E \equiv \int E_\nu d\nu, \quad F^i \equiv \int F^i_\nu d\nu, \quad P^{ij} \equiv \int P^{ij}_\nu d\nu. \tag{11.24}$$

(f) Transformation rules

The transformation of the frequency-integrated intensity I between the inertial and comoving frames is

$$I_0 = \left(\frac{\nu_0}{\nu}\right)^4 I = \left[\gamma\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)\right]^4 I. \tag{11.25}$$

Integrating equation (3.25) over a solid angle, we obtain the transformation rule of E:

$$E_0 = \gamma^2\left(E - 2\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2} + \frac{v_i v_k}{c^2}P^{ik}\right). \tag{11.26}$$

Multiplying equation (3.25) by l_0^i and integrating the resultant equation over a solid angle, we have the transformation rule of F^i:

$$F_0^i = \gamma \left\{ F^i + \left[\left(\gamma + \frac{\gamma-1}{v^2/c^2} \right) \frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2} - \gamma E - \frac{\gamma-1}{v^2/c^2}\frac{v_j v_k}{c^2} P^{jk} \right] v^i - v_k P^{ik} \right\}. \tag{11.27}$$

Multiplying equation (3.25) by $l_0^i l_0^j$ and integrating the resultant equation over a solid angle, we have the transformation rule of P^{ij}:

$$\begin{aligned} P_0^{ij} &= P^{ij} + \frac{\gamma-1}{v^2/c^2}\left(\frac{v^i v_k}{c^2} P^{jk} + \frac{v^j v_k}{c^2} P^{ik} \right) \\ &+ \left(\frac{\gamma-1}{v^2/c^2} \right)^2 \frac{v^i v^j}{c^2} \frac{v_k v_m P^{km}}{c^2} + \gamma^2 \frac{v^i v^j}{c^2} E \\ &- \gamma \left(\frac{v^i F^j}{c^2} + \frac{v^j F^i}{c^2} \right) - 2\gamma \frac{\gamma-1}{v^2/c^2} \frac{v^i v^j}{c^2} \frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2}. \end{aligned} \tag{11.28}$$

(g) **Energy-momentum tensor**

The energy-momentum tensor for an ideal gas, $T^{\mu\nu}$, is

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu}, \tag{11.29}$$

where ε is the internal energy per unit proper volume and p is the pressure measured in the comoving frame ($\varepsilon + p$ is the enthalpy per unit proper volume).

The energy-momentum tensor for radiation, $R^{\mu\nu}$, is

$$R^{\mu\nu} = \begin{pmatrix} E & \frac{1}{c}F^i \\ \frac{1}{c}F^i & P^{ij} \end{pmatrix}, \tag{11.30}$$

where E is the radiation energy density, F^i the radiative flux, and P^{ij} the radiation stress tensor.

The momentum and energy conservations are expressed, respectively, as

$$\left(T_\mu{}^\nu + R_\mu{}^\nu \right)_{;\nu} = 0, \tag{11.31}$$

$$u^\mu \left(T_\mu{}^\nu + R_\mu{}^\nu \right)_{;\nu} = 0, \tag{11.32}$$

where the semicolon means the partial differentiation in the present case.

11.2 Equations of Radiative Transfer

We first derive the basic equations describing the behavior of radiation interacting with matter within the framework of special relativity.

11.2.1 Transfer Equation

As in the case of a non-relativistic regime (appendix D), a change in the specific intensity is expressed by the *transfer equation*, although it should be written down in a Lorentz-invariant form.

By means of the Lorentz invariant f $(= I_\nu/\nu^3 = I_{\nu 0}/\nu_0^3)$, we can write the transfer equation of the form (Hsieh and Spiegel 1976):

$$\begin{aligned} k^\mu \frac{\partial f}{\partial x^\mu} &= \rho\,(\alpha - \beta f) - \rho \kappa_{\nu 0}^{\rm sca} \int \phi_\nu(\boldsymbol{l}', \boldsymbol{l}) f(\boldsymbol{l}) \nu' d\nu' d\Omega' \\ &\quad + \rho \kappa_{\nu 0}^{\rm sca} \int \phi_\nu(\boldsymbol{l}, \boldsymbol{l}') f(\boldsymbol{l}') \nu' d\nu' d\Omega', \end{aligned} \tag{11.33}$$

where ρ is the proper mass density, α the invariant form of the emission coefficient, β the invariant form of the absorption coefficient, $\kappa_{\nu 0}^{\rm sca}$ the scattering opacity in the comoving frame, and ϕ_ν the scattering redistribution function. It is noted that $\nu d\nu d\Omega$ $(= \nu' d\nu' d\Omega')$ is also a relativistic invariant.

Of these, α and β are related, respectively, to the mass emissivity $j_{\nu 0}$ and the mass absorption coefficient $\kappa_{\nu 0}^{\rm abs}$ in the comoving frame by

$$j_{\nu 0} = 4\pi \nu_0^2 \alpha \quad \text{and} \quad \kappa_{\nu 0}^{\rm abs} = \frac{\beta}{\nu_0}. \tag{11.34}$$

For Thomson scattering, the scattering redistribution function in the comoving frame is

$$\phi_\nu = \frac{3}{4} \left[1 + \left(\boldsymbol{l}_0 \cdot \boldsymbol{l}_0' \right)^2 \right] \delta(\nu_0 - \nu_0') \frac{1}{4\pi}. \tag{11.35}$$

It should be noted that $\int \phi_\nu \nu_0 d\nu_0 d\Omega_0 = \nu_0'$ and $\int \phi_\nu \nu_0' d\nu_0' d\Omega_0' = \nu_0$.

Substituting these quantities into equation (3.33), the transfer equation is rewritten as

$$\nu \left[\frac{\partial f}{c \partial t} + (\boldsymbol{l} \cdot \boldsymbol{\nabla})\, f \right] = \rho \frac{j_{\nu 0}}{4\pi \nu_0^2} - \rho \nu_0 \kappa_{\nu 0}^{\rm abs} f - \rho \nu_0 \kappa_{\nu 0}^{\rm sca} f$$

$$+\frac{3}{4}\rho\kappa_{\nu 0}^{\rm sca}\nu_0 \int \left[1 + (\boldsymbol{l}_0 \cdot \boldsymbol{l}'_0)^2\right] f(\boldsymbol{l}') \frac{d\Omega'_0}{4\pi}. \qquad (11.36)$$

Furthermore, replacing f by I_ν (or $I_{\nu 0}$), we finally obtain the (angle-dependent) *radiative transfer equation* in the mixed frame:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu = \left(\frac{\nu}{\nu_0}\right)^2 \rho$$
$$\times \left[\frac{j_{\nu 0}}{4\pi} - \left(\kappa_{\nu 0}^{\rm abs} + \kappa_{\nu 0}^{\rm sca}\right) I_{\nu 0} + \frac{3}{4}\kappa_{\nu 0}^{\rm sca}\frac{c}{4\pi}\left(E_{\nu 0} + l_{0i}l_{0j}P_{\nu 0}^{ij}\right)\right], \qquad (11.37)$$

where we use the definitions of E and P^{ij}. This transfer equation (3.37) seems to be similar to the non-relativistic one (D.7), except for the ν/ν_0-term. It should be noted, however, that the left-hand side is written by the quantities evaluated in the inertial (fixed) frame, while the right-hand side by the quantities in the comoving (fluid) frame.

11.2.2 Moment Equations

Next, we derive the (frequency-integrated) moment equations. After a long time since Eddington, who first introduced a moment expansion to radiation transfer in the early 20th century, moment equations for relativistic radiation transfer have been derived by several studies for a special relativistic case (Thomas 1930; Hazlehurst and Sargent 1959; Castor 1972; Mihalas and Mihalas 1984) and in a curved space-time (Lindquist 1966; Anderson and Spiegel 1972; Thorne 1981; Udey and Israel 1982; Nobili et al. 1993; Park 2003, 2006; Takahashi 2007). A complete set of moment equations for a relativistic flow is given by the projected symmetric trace-free (PSTF) formalism (Thorne 1981).

Integrating the transfer equation (3.37) over the frequency, with the help of the Lorentz transformation (3.14) [$d\nu = (d\nu/d\nu_0)d\nu_0 = \gamma(1 + \boldsymbol{v}\cdot\boldsymbol{l}_0/c)d\nu_0$], we obtain a frequency-integrated angle-dependent transfer equation:

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I = \rho\gamma^3 \left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{l}_0}{c}\right)^3$$
$$\times \left[\frac{j_0}{4\pi} - \left(\kappa_0^{\rm abs} + \kappa_0^{\rm sca}\right) I_0 + \frac{3}{4}\kappa_0^{\rm sca}\frac{c}{4\pi}\left(E_0 + l_{0i}l_{0j}P_0^{ij}\right)\right], \qquad (11.38)$$

where

$$I \equiv \int I_\nu d\nu, \quad I_0 \equiv \int I_{\nu 0} d\nu_0, \qquad (11.39)$$

$$E_0 \equiv \int E_{\nu 0} d\nu_0, \quad P_0^{ij} \equiv \int P_{\nu 0}^{ij} d\nu_0, \qquad (11.40)$$

$$j_0 \equiv \int j_{\nu 0} d\nu_0, \quad \kappa_0^{\rm abs} + \kappa_0^{\rm sca} \equiv \frac{1}{I_0}\int \left(\kappa_{\nu 0}^{\rm abs} + \kappa_{\nu 0}^{\rm sca}\right) I_{\nu 0} d\nu_0. \qquad (11.41)$$

Integrating the transfer equation (3.38) over a solid angle, with the help of a transformation of the solid angle (3.18), we obtain the zeroth-moment of equation (3.38):

$$\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} = \rho\gamma\left(j_0 - c\kappa_0^{\rm abs}E_0\right) - \rho\gamma\left(\kappa_0^{\rm abs} + \kappa_0^{\rm sca}\right)\frac{\boldsymbol{v}\cdot\boldsymbol{F}_0}{c}. \qquad (11.42)$$

Integrating the transfer equation (3.38) over a solid angle, after being multiplied by the direction cosine, with the help of transformations (3.18) and (3.16), we obtain the first-moment of equation (3.38):

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} = \rho\gamma\frac{v^i}{c^2}\left(j_0 - c\kappa_0^{\rm abs}E_0\right)$$
$$-\rho\left(\kappa_0^{\rm abs} + \kappa_0^{\rm sca}\right)\frac{\gamma - 1}{v^2}\frac{v^i}{c}\left(\boldsymbol{v}\cdot\boldsymbol{F}_0\right)$$
$$-\frac{1}{c}\rho\left(\kappa_0^{\rm abs} + \kappa_0^{\rm sca}\right)F_0^i. \qquad (11.43)$$

In general, such a moment expansion gives an infinite set of equations. In order to make the transfer problem tractable, one must truncate the expansion at the finite order by adopting a suitable closure assumption. For example, we here truncate the equations at the second order, and we introduce some additional closure relation among E, F^i, and P^{ik}, as given in the next subsection.

As already noted, the left-hand sides of these moment equations (3.42) and (3.43) are described by the quantities in the inertial (fixed) frame, while the right-hand sides by those in the comoving (fluid) frame. Thus, using the transformation rules (3.26)–(3.28), let us rewrite the right-hands side of these moment equations. After several manipulations, we finally obtain the moment equations expressed by the quantities in the inertial (fixed) frame:

$$\begin{aligned}\frac{1}{c}\frac{\partial I}{\partial t}+(\boldsymbol{l}\cdot\boldsymbol{\nabla})\,I&=\rho\gamma^{-3}\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^{-3}\\&\times\left[\frac{j_0}{4\pi}-\left(\kappa_0^{\mathrm{abs}}+\kappa_0^{\mathrm{sca}}\right)\gamma^4\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^4 I+\frac{\kappa_0^{\mathrm{sca}}}{4\pi}\frac{3}{4}\gamma^{-2}\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^{-2}\right.\\&\times\left\{\gamma^4\left[\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^2+\left(\frac{v^2}{c^2}-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^2\right]cE+2\gamma^2\left(\frac{v^2}{c^2}-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)\boldsymbol{F}\cdot\boldsymbol{l}\right.\\&-2\gamma^4\left[\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^2+\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)\left(\frac{v^2}{c^2}-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)\right]\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c}\\&\left.\left.+l_il_jcP^{ij}-2\gamma^2\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)v_il_jP^{ij}+2\gamma^4\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{l}}{c}\right)^2\frac{v_iv_jP^{ij}}{c}\right\}\right],\end{aligned}\tag{11.44}$$

$$\begin{aligned}\frac{\partial E}{\partial t}+\frac{\partial F^k}{\partial x^k}&=\rho\gamma\left(j_0-c\kappa_0^{\mathrm{abs}}E+\kappa_0^{\mathrm{abs}}\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c}\right)\\&+\rho\gamma^3\kappa_0^{\mathrm{sca}}\left[\frac{v^2}{c}E+\frac{v_iv_j}{c}P^{ij}-\left(1+\frac{v^2}{c^2}\right)\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c}\right],\end{aligned}\tag{11.45}$$

$$\begin{aligned}\frac{1}{c^2}\frac{\partial F^i}{\partial t}+\frac{\partial P^{ik}}{\partial x^k}&=\frac{\rho\gamma}{c}\left(\frac{v^i}{c}j_0-\kappa_0^{\mathrm{abs}}F^i+\kappa_0^{\mathrm{abs}}v_kP^{ik}\right)\\&-\frac{\rho\gamma}{c}\kappa_0^{\mathrm{sca}}\left[F^i-\gamma^2Ev^i-v_kP^{ik}+\gamma^2v^i\left(\frac{2\boldsymbol{v}\cdot\boldsymbol{F}}{c^2}-\frac{v_jv_k}{c^2}P^{jk}\right)\right].\end{aligned}\tag{11.46}$$

11.2.3 Closure Relation

As a closure relation, we usually adopt the Eddington approximation *in the comoving frame*:

$$P_0^{ij}=\frac{\delta^{ij}}{3}E_0.\tag{11.47}$$

It should be noted that we here do not consider the radiative viscosity. In a relativistic regime, this closure relation should be modified, as discussed in the next subsection.

Substituting the transformation rules (3.26)–(3.28) into this relation (3.47), we obtain the closure relation in the inertial frame:

$$\begin{aligned}P^{ij}&-\frac{\delta^{ij}}{3}\gamma^2\frac{v_kv_m}{c^2}P^{km}+\frac{\gamma^2}{\gamma+1}\left(\frac{v^iv_k}{c^2}P^{jk}+\frac{v^jv_k}{c^2}P^{ik}\right)\\&+\left(\frac{\gamma^2}{\gamma+1}\right)^2\frac{v^iv^j}{c^2}\frac{v_kv_m}{c^2}P^{km}=\frac{\delta^{ij}}{3}\gamma^2\left(E-2\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2}\right)-\gamma^2\frac{v^iv^j}{c^2}E\\&+\gamma\left(\frac{v^iF^j}{c^2}+\frac{v^jF^i}{c^2}\right)+2\gamma\frac{\gamma^2}{\gamma+1}\frac{v^iv^j}{c^2}\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2}.\end{aligned}\tag{11.48}$$

To the first order in $\boldsymbol{v}/c$, the closure relation becomes (Hsieh and Spiegel 1976)

$$P^{ij}=\frac{\delta^{ij}}{3}E+\frac{v^iF^j}{c^2}+\frac{v^jF^i}{c^2}-\frac{2}{3}\delta^{ij}\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{c^2}.\tag{11.49}$$

11.3 Relativistic Regimes

The radiation moment formalism is quite convenient and essential, especially in a relativistic regime, and it is a powerful tool for tackling problems of relativistic radiation hydrodynamics (e.g., Thorne et al. 1981; Flammang 1982, 1984; Nobili et al. 1991, 1993; Park 2001, 2006 for spherically symmetric problems; Takahashi 2007 for the Kerr metric). However, its validity is never known unless a fully angle-dependent radiation transfer equation is solved. Thus, the relativistic moment equations with a closure relation must be carefully treated, and applied to black-hole accretion flows, relativistic jets and winds, and relativistic explosions, such as gamma-ray bursts.

Actually, the pathological behavior in relativistic radiation moment equations has been pointed out and examined (Turolla and Nobili 1988; Nobili et al. 1991; Turolla et al. 1995; Dullemond 1999). Namely, the moment equations for radiation transfer in relativistically moving media can generally have singular (critical) points. As a result, solutions behave pathologically in a relativistic regime. The appearance of singularities is supposed to be related to the approximation of the full transfer equations with a finite number of moments (Dullemond 1999).

For example, in one-dimensional relativistic flows using the closure relation (3.47), where the moment equations are truncated at the second order, the singularity appears when the flow velocity v becomes $\pm c/\sqrt{3}$, which corresponds to the relativistic sound speed (Turolla and Nobili 1988; Turolla et al. 1995). Hence, we cannot obtain solutions accelerated beyond $c/\sqrt{3}$, altough there exists a decelerating solution (Fukue 2005).

The invalidity of the Eddington approximation in such a relativistic flow can be understood as follows. In adopting the Eddington approximation (3.47), we assume that within the photon mean-free path the radiation field is *isotropic* in the comoving frame. However, in the relativistic regime, where the velocity gradient becomes large and there exist the Doppler and aberration effects of photons, the isotropy of the radiation field may break down even in the comoving frame.

For example, the photon mean-free path ℓ in the comoving frame is $\ell \sim 1/(\kappa\rho)$, where κ is the opacity measured in the comoving frame and ρ is the proper density. When there exists a (strong) velocity gradient, say dv/ds, the velocity increase at the distance of ℓ is estimated as

$$\Delta v = \ell \frac{dv}{ds} = \frac{dv}{\kappa\rho ds} = \frac{dv}{d\tau}, \tag{11.50}$$

where τ $(= \kappa\rho s)$ is the optical depth. In order for the radiation fields to be isotropic in the comoving frame, this velocity increase should be sufficiently smaller than the speed of light. In such a case, we should modify the closure relation in the case of subrelativistic to relativistic regimes, as in the case of optically thick to thin regimes.

11.3.1 Velocity-Dependent Variable Eddington Factor

In order to improve the situation we are confronted with, instead of the usual Eddington approximation, we can adopt a *variable Eddington factor*, which depends on the flow velocity β $(= v/c)$ and the velocity gradient $d\beta/d\tau$ as well as the optical depth τ (Fukue 2006; Akizuki and Fukue 2007; Fukue 2007b; Koizumi and Umemura 2007). In one-dimensional flows the variable Eddington factor $f(\tau, \beta)$ is generally defined as

$$P_0 = f(\tau, \beta) E_0, \tag{11.51}$$

where E_0 and P_0 are the radiation energy density and the radiation stress tensor in the comoving frame, respectively. The closure relation in the inertial frame for one-dimensional flows then becomes

$$cP(1 - f\beta^2) = cE(f - \beta^2) + 2F\beta(1 - f), \tag{11.52}$$

where E, F, and P are the radiation energy density, the radiative flux, and the radiation pressure in the inertial frame, respectively.

The function $f(\tau, \beta)$ must reduce to $1/3$ or appropriate values in the non-relativistic limit of $\beta = 0$, whereas it would approach unity in the extremely relativistic limit of $\beta = 1$. Furthermore, in the sufficiently optically thick regime this function approaches $1/3$ except for $\beta = 1$, while in the optically thin limit it reduces to an appropriate form determined by the geometry under the considerations.[1] The appropriate form is now under construction.

11.4 Matter Coupling

We can now write the basic equations for matter (Hsieh and Spiegel 1976; Fukue et al. 1985; Park 2006 for the Schwarzschild metric; Takahashi 2007 for the Kerr metric).

(a) **Mass conservation**

The particle number conservation is

$$(nu^\mu)_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} n u^\mu \right) = 0, \tag{11.53}$$

[1] In the plane-parallel case, for instance, the variable Eddinton factor in the optically thin limit is analytically derived as

$$f = \frac{1 - 3\beta + 3\beta^2}{3 - 3\beta + \beta^2}.$$

where x^μ is the space-time coordinates, u^μ the four-velocity, and n the proper number density.

In the three-dimensional form, the mass conservation becomes

$$\frac{\partial}{\partial t}(\rho\gamma) + \mathrm{div}(\rho\gamma\boldsymbol{v}) = 0, \tag{11.54}$$

where ρ $(= nmc^2)$ is the proper density.

(b) **Momentum conservation**

The relativistic equations of motion, $(T_i^{\ \mu} + R_i^{\ \mu})_{;\mu} = 0$, are written as

$$(\varepsilon + p)\left(u^\mu \frac{\partial u^i}{\partial x^\mu} + \Gamma^i_{\mu\nu}u^\mu u^\nu\right) - \left(g^{i\mu} - u^i u^\mu\right)\frac{\partial p}{\partial x^\mu} = -(g^{i\mu} - u^i u^\mu) R_{\mu\ ;\nu}^{\ \nu}, \tag{11.55}$$

where ε is the internal energy per unit proper volume, p the pressure measured in the comoving frame, $T^{\mu\nu}$ the energy-momentum tensor of matter, and $R^{\mu\nu}$ the stress-energy tensor of radiation.

The right-hand side of equation (3.55) are, from (3.30), (3.42), (3.43), (3.45) and (3.46),

$$\begin{aligned}
&-\left(g^{i\mu} - u^i u^\mu\right) R_{\mu\ ;\nu}^{\ \nu} \\
&= -\left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k}\right) \\
&\quad -\frac{\gamma^2}{c^2}v^i\left[-\left(\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k}\right) + v_j\left(\frac{1}{c^2}\frac{\partial F^j}{\partial t} + \frac{\partial P^{jk}}{\partial x^k}\right)\right] \\
&= \frac{\rho}{c}\left(\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}\right)\left[F_0^i + \frac{\gamma-1}{v^2}v^i(\boldsymbol{v}\cdot\boldsymbol{F}_0)\right] \\
&= \frac{\rho\gamma}{c}\left(\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}\right) \\
&\quad \times\left[F^i - \gamma^2 E v^i - v_k P^{ik} + \gamma^2 v^i\left(\frac{2\boldsymbol{v}\cdot\boldsymbol{F}}{c^2} - \frac{v_j v_k}{c^2}P^{jk}\right)\right].
\end{aligned} \tag{11.56}$$

Thus, the relativistic equations of motion are

$$\begin{aligned}
&c^2\left(u^\mu \frac{\partial u^i}{\partial x^\mu} + \Gamma^i_{\mu\nu}u^\mu u^\nu\right) \\
&= \frac{c^2}{\varepsilon+p}\left(g^{i\mu} - u^i u^\mu\right)\frac{\partial p}{\partial x^\mu} + \frac{\rho c^2}{\varepsilon + p}\frac{1}{c}\left(\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}\right) \\
&\quad \times\left[F_0^i + \frac{\gamma-1}{v^2}v^i(\boldsymbol{v}\cdot\boldsymbol{F}_0)\right] \\
&= \frac{c^2}{\varepsilon+p}\left(g^{i\mu} - u^i u^\mu\right)\frac{\partial p}{\partial x^\mu} + \frac{\rho c^2}{\varepsilon + p}\frac{\gamma}{c}\left(\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}\right) \\
&\quad \times\left[F^i - \gamma^2 E v^i - v_k P^{ik} + \gamma^2 v^i\left(\frac{2\boldsymbol{v}\cdot\boldsymbol{F}}{c^2} - \frac{v_j v_k}{c^2}P^{jk}\right)\right].
\end{aligned} \tag{11.57}$$

(c) **Energy conservation**

The energy conservation, $u^\mu(T_\mu^{\ \nu} + R_\mu^{\ \nu})_{;\nu} = 0$, is written as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\varepsilon u^\mu\right) + \frac{p}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}u^\mu\right) = -u^\mu R_{\mu\ ;\nu}^{\ \nu}. \tag{11.58}$$

The right-hand side of equation (3.58) is, from (3.30), (3.42), (3.43), (3.45) and (3.46),

$$\begin{aligned}
-u^\mu R_{\mu\ ;\nu}^{\ \nu} &= -\frac{\gamma}{c}\left(\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k}\right) + \frac{\gamma v_i}{c}\left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k}\right) \\
&= -\frac{\rho}{c}\left(j_0 - c\kappa_0^{\mathrm{abs}}E_0\right) \\
&= \frac{\gamma^2\rho}{c}\left(-\frac{j_0}{\gamma^2} + c\kappa_0^{\mathrm{abs}}E - \kappa_0^{\mathrm{abs}}\frac{2\boldsymbol{v}\cdot\boldsymbol{F}}{c} + \kappa_0^{\mathrm{abs}}\frac{v_i v_k}{c}P^{ik}\right).
\end{aligned} \tag{11.59}$$

Thus, the energy equation is

$$\begin{aligned}
&\frac{c}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\left(\varepsilon - \rho c^2\right)u^\mu\right] + c\frac{p}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}u^\mu\right) \\
&= -\rho\left(j_0 - c\kappa_0^{\mathrm{abs}}E_0\right) \\
&= \gamma^2\rho\left(-\frac{j_0}{\gamma^2} + c\kappa_0^{\mathrm{abs}}E - \kappa_0^{\mathrm{abs}}\frac{2\boldsymbol{v}\cdot\boldsymbol{F}}{c} + \kappa_0^{\mathrm{abs}}\frac{v_i v_k}{c}P^{ik}\right).
\end{aligned} \tag{11.60}$$

(d) **Sub-relativistic Regime**

To the first order of $(\boldsymbol{v}/c)$, the equations of motion and energy equation for matter are written as, respectively,

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} &= -\boldsymbol{\nabla}\psi - \frac{1}{\rho}\boldsymbol{\nabla} p \\ &+ \frac{\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}}{c} \left(\boldsymbol{F} - E\boldsymbol{v} - v_k P^{ik} \right), \end{aligned} \quad (11.61)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) e + \frac{p}{\rho} \boldsymbol{\nabla v} = \frac{1}{\rho} q^+ - j_0 + c\kappa_0^{\mathrm{abs}} E - \kappa_0^{\mathrm{abs}} \frac{\boldsymbol{2v \cdot F}}{c}, \quad (11.62)$$

where $\boldsymbol{v}$ is the velocity, ψ the gravitational potential, p the pressure, e the internal energy per unit mass, and q^+ the (viscous) heating rate per unit volume (Hsieh and Spiegel 1976; Fukue et al. 1985).

The equations for radiation are, on the other hand,

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla F} = \rho \left[j_0 - c\kappa_0^{\mathrm{abs}} E + \left(\kappa_0^{\mathrm{abs}} - \kappa_0^{\mathrm{sca}} \right) \frac{\boldsymbol{v} \cdot \boldsymbol{F}}{c} \right], \quad (11.63)$$

$$\begin{aligned} \frac{1}{c^2} \frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} &= \frac{\rho}{c} \left(\frac{v^i}{c} j_0 - \kappa_0^{\mathrm{abs}} F^i + \kappa_0^{\mathrm{abs}} v_k P^{ik} \right) \\ &- \frac{\rho}{c} \kappa_0^{\mathrm{sca}} \left(F^i - E v^i - v_k P^{ik} \right), \end{aligned} \quad (11.64)$$

$$P^{ij} = \frac{\delta^{ij}}{3} E + \frac{v^i F^j}{c^2} + \frac{v^j F^i}{c^2} - \frac{2}{3} \frac{\boldsymbol{v} \cdot \boldsymbol{F}}{c^2} \delta^{ij}. \quad (11.65)$$

11.5 Plane-Parallel Expression

For a relativistically moving atmosphere in the plane-parallel geometry (z), the hydrodynamic equations and transfer equations become as follows (Fukue 2006, 2007a, b).

For matter, the continuity equation is

$$\rho c u = \rho \gamma \beta c = J \ (= \text{const.}), \quad (11.66)$$

where ρ is the proper gas density, u the vertical four velocity, J the mass-loss rate per unit area, and c the speed of light. The four velocity u is related to the proper three velocity v by $u = \gamma\beta = \gamma v/c$, where γ is the Lorentz factor, $\gamma = \sqrt{1+u^2} = 1/\sqrt{1-(v/c)^2}$.

The equation of motion is

$$\begin{aligned} c^2 u \frac{du}{dz} &= c^2 \gamma^4 \beta \frac{d\beta}{dz} \\ &= -\frac{d\psi}{dz} - \gamma^2 \frac{c^2}{\varepsilon + p} \frac{dp}{dz} \\ &+ \frac{\rho c^2}{\varepsilon + p} \frac{\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}}}{c} \gamma^3 \left[F(1+\beta^2) - (cE + cP)\beta \right], \end{aligned} \quad (11.67)$$

where ψ is the gravitational potential, p the gas pressure, κ_0^{abs} and κ_0^{sca} are the absorption and scattering opacities (gray), defined in the comoving frame, E the radiation energy density, F the radiative flux, and P the radiation pressure observed in the inertial frame. The first term in the square brackets on the right-hand side of equation (3.67) means the radiatively-driven force, which is modified to the order of u^2, whereas the second term is the radiation drag force, which is also modified, but roughly proportional to the velocity.

When the gas pressure is ignored, the advection terms of the energy equation are dropped, and heating is balanced with the radiative terms,

$$0 = \frac{q^+}{\rho} - \left(j_0 - \kappa_0^{\mathrm{abs}} cE\gamma^2 - \kappa_0^{\mathrm{abs}} cPu^2 + 2\kappa_0^{\mathrm{abs}} F\gamma u \right), \quad (11.68)$$

where q^+ is the internal heating and j_0 is the emissivity defined in the comoving frame. In this equation (3.68), the third and fourth terms on the right-hand side appear in the relativistic regime. Under the α prescription, the viscous-heating rate is proportional to the pressure, and therefore, may depend on z.

For radiation, the frequency-integrated transfer equation (3.44), the zeroth moment equation (3.45), and the first moment equation (3.46) become, respectively:

$$\begin{aligned} \mu \frac{dI}{dz} &= \rho \frac{1}{\gamma^3 (1-\beta\mu)^3} \left[\frac{j_0}{4\pi} - \left(\kappa_0^{\mathrm{abs}} + \kappa_0^{\mathrm{sca}} \right) \gamma^4 (1-\beta\mu)^4 I \right. \\ &+ \frac{\kappa_0^{\mathrm{sca}}}{4\pi} \frac{3}{4} \gamma^2 \left\{ \left[1 + \frac{(\mu-\beta)^2}{(1-\beta\mu)^2} \beta^2 + \frac{(1-\beta^2)^2}{(1-\beta\mu)^2} \frac{1-\mu^2}{2} \right] cE \right. \end{aligned}$$

$$
\begin{aligned}
& -\left[1+\frac{(\mu-\beta)^2}{(1-\beta\mu)^2}\right]2F\beta \\
& +\left[\beta^2+\frac{(\mu-\beta)^2}{(1-\beta\mu)^2}-\frac{(1-\beta^2)^2}{(1-\beta\mu)^2}\frac{1-\mu^2}{2}\right]cP\Bigg\}\Bigg], \qquad (11.69)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF}{dz} &= \rho\gamma\Big[\, j_0-\kappa_0^{\mathrm{abs}}cE+\kappa_0^{\mathrm{sca}}(cE+cP)\gamma^2\beta^2 \\
&\quad +\kappa_0^{\mathrm{abs}}F\beta-\kappa_0^{\mathrm{sca}}F(1+\beta^2)\gamma^2\beta\,\Big]. \qquad (11.70)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dP}{dz} &= \frac{\rho\gamma}{c}\Big[\, j_0\beta-\kappa_0^{\mathrm{abs}}F+\kappa_0^{\mathrm{abs}}cP\beta \\
&\quad -\kappa_0^{\mathrm{sca}}F\gamma^2(1+\beta^2)+\kappa_0^{\mathrm{sca}}(cE+cP)\gamma^2\beta\,\Big], \qquad (11.71)
\end{aligned}
$$

where $\mu=\cos\theta$.

Finally, a closure relation is

$$
cP(1-f\beta^2)=cE(f-\beta^2)+2F\beta(1-f), \qquad (11.72)
$$

where $f(\tau,\beta)$ is the variable Eddington factor depending on the velocity as well as the optical depth.

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