第23章 Relativistic Radiative Transfer Plane-Parallel Comoving Frame variables $I_0 E_0 F_0 P_0$ in physics

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23.1 Relativistic Radiative Transfer Equation

The radiative transfer equations are given in several literatures (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 1998, 2008; Mihalas, Auer 2001; Peraiah 2002; Castor 2004). The basic equations for relativistic radiation hydrodynamics are given in, e.g., the appendix E of Kato et al. (2008) in general and vertical forms (See also Fukue 2008c).

23.1.1 General Form

In a general form the radiative transfer equation in the *mixed frame*, where the variables in the inertial and comoving frames are used, is expressed as

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I = \left(\frac{\nu}{\nu_0}\right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0 \frac{cE_0}{4\pi}\right], \quad (23.1)$$

where c is the speed of light. In the left-hand side the frequency-integrated specific intensity I and the direction cosine $\boldsymbol{l}$ are quantities measured in the inertial (fixed) frame. In the right-hand side, on the other hand, the mass density ρ, the frequency-integrated mass emissivity j_0, the frequency-integrated mass absorption coefficient κ_0, the frequency-integrated mass scattering coefficient σ_0, the frequency-integrated specific intensity I_0, and the frequency-integrated radiation energy density E_0 are quantities measured in the comoving (fluid) frame. In this paper, instead of the weakly anisotropic Thomson scattering, we assume that the scattering is isotropic for simplicity.

The Doppler effect, the aberration, and the transformation of the intensities are expressed as

$$\frac{\nu}{\nu_0} = \gamma\,(1+\boldsymbol{\beta}\cdot\boldsymbol{l}_0), \quad (23.2)$$

$$\boldsymbol{l} = \frac{\nu_0}{\nu}\left[\boldsymbol{l}_0 + \left(\frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\cdot\boldsymbol{l}_0 + \gamma\right)\boldsymbol{\beta}\right], \quad (23.3)$$

$$I = \left(\frac{\nu}{\nu_0}\right)^4 I_0, \quad (23.4)$$

where ν and ν_0 are the frequencies measured in the inertial and comoving frames, respectively, the direction cosine $\boldsymbol{l}_0$ measured in the comoving frame, $\boldsymbol{\beta}$ $(=\boldsymbol{v}/c)$ the normalized velocity, $\boldsymbol{v}$ being the flow velocity, and γ $(=1/\sqrt{1-\beta^2})$ the Lorentz factor, β being v/c.

The zeroth and first moment equations are, respectively,

$$\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} = \rho\gamma\,(j_0 - \kappa_0 c E_0) - \rho\gamma\,(\kappa_0+\sigma_0)\,\boldsymbol{\beta}\cdot\boldsymbol{F}_0, \quad (23.5)$$

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} = \rho\gamma\frac{\beta^i}{c}\,(j_0 - \kappa_0 c E_0) - \rho\,(\kappa_0+\sigma_0)\,\frac{\gamma-1}{\beta^2}\frac{\beta^i}{c}\,(\boldsymbol{\beta}\cdot\boldsymbol{F}_0) - \frac{1}{c}\rho\,(\kappa_0+\sigma_0)\,F_0^i, \quad (23.6)$$

where the frequency-integrated radiation energy density E, the frequency-integrated radiative flux $\boldsymbol{F}$, and the frequency-integrated radiation stress P^{ik} are measured in the inertial frame, while those with the subscript 0 are measured in the comoving frame.

As a closure relation, we adopt the Eddington approximation *in the comoving frame*:

$$P_0^{ik} = f^{ik} E_0, \tag{23.7}$$

where f^{ik} is the Eddington tensor, which is generally a function of the optical depth and flow speed in the relativistic radiative flow.

23.1.2 Plane-Parallel Expression in the Comoving Frame

Let us suppose a relativistic flow in the vertical direction. In the plane-parallel geometry with the vertical axis z the transfer equation (23.1) is expressed as

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} = \left(\frac{\nu}{\nu_0}\right)^3 \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0\frac{cE_0}{4\pi}\right], \tag{23.8}$$

where μ is the direction cosine in the inertial frame. Inserting the transformation (23.4) in the left-hand side, this equation (23.8) becomes

$$\begin{aligned}
&\frac{\nu}{\nu_0}\left(\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial z}\right) - 4\frac{\nu}{\nu_0^2} I_0 \left(\frac{1}{c}\frac{\partial \nu_0}{\partial t} + \mu\frac{\partial \nu_0}{\partial z}\right) \\
&= \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0\frac{cE_0}{4\pi}\right].
\end{aligned} \tag{23.9}$$

To calculate the derivatives of I_0 (Mihalas, Mihalas 1984), we apply the chain rules and after some manipulations we have

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{z\mu\nu} &= \left.\frac{\partial}{\partial t}\right|_{z\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial t}\right|_{z\mu_0\nu_0}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial t}\right|_{z\mu_0\nu_0}\frac{\partial}{\partial \nu_0} \\
&= \left.\frac{\partial}{\partial t}\right|_{z\mu_0\nu_0} - \gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial t}\frac{\partial}{\partial \mu_0} - \gamma^2\mu_0\nu_0\frac{\partial \beta}{\partial t}\frac{\partial}{\partial \nu_0},
\end{aligned} \tag{23.10}$$

$$\begin{aligned}
\left.\frac{\partial}{\partial z}\right|_{t\mu\nu} &= \left.\frac{\partial}{\partial z}\right|_{t\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial z}\right|_{t\mu_0\nu_0}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial z}\right|_{t\mu_0\nu_0}\frac{\partial}{\partial \nu_0} \\
&= \left.\frac{\partial}{\partial z}\right|_{t\mu_0\nu_0} - \gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial z}\frac{\partial}{\partial \mu_0} - \gamma^2\mu_0\nu_0\frac{\partial \beta}{\partial z}\frac{\partial}{\partial \nu_0},
\end{aligned} \tag{23.11}$$

where μ_0 is the direction cosine in the comoving frame. In addition, the Doppler shift (23.2) and the aberration (23.3) are respectively expressed as

$$\frac{\nu}{\nu_0} = \gamma(1+\beta\mu_0), \tag{23.12}$$

$$\mu = \frac{\mu_0+\beta}{1+\beta\mu_0}. \tag{23.13}$$

Using these expressions, after some manipulations we have the radiative transfer equation in the comoving frame for the plane-parallel flow:

$$\begin{aligned}
&\gamma(1+\beta\mu_0)\frac{1}{c}\frac{\partial I_0}{\partial t} + \gamma(\mu_0+\beta)\frac{\partial I_0}{\partial z} \\
&-\gamma^3(1+\beta\mu_0)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{1}{c}\frac{\partial \beta}{\partial t} \\
&-\gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{\partial \beta}{\partial z} \\
&= \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0\frac{cE_0}{4\pi}\right].
\end{aligned} \tag{23.14}$$

Integrating the transfer equation (23.14) over a solid angle, we have the zeroth and first moment equations in the comoving frame for the plane-parallel flow:

$$\begin{aligned}
&\gamma\frac{\partial cE_0}{c\partial t} + \gamma\frac{\partial F_0}{\partial z} + \gamma\beta\frac{\partial F_0}{c\partial t} + \gamma\beta\frac{\partial cE_0}{\partial z} \\
&+\gamma^3\left[2F_0 + \beta(cE_0+cP_0)\right]\frac{\partial \beta}{c\partial t} \\
&+\gamma^3\left[2\beta F_0 + (cE_0+cP_0)\right]\frac{\partial \beta}{\partial z} \\
&= \rho\left(j_0 - \kappa_0 cE_0\right),
\end{aligned} \tag{23.15}$$

$$\begin{aligned}
&\gamma\frac{\partial F_0}{c\partial t} + \gamma\frac{\partial cP_0}{\partial z} + \gamma\beta\frac{\partial cP_0}{c\partial t} + \gamma\beta\frac{\partial F_0}{\partial z} \\
&+\gamma^3\left[2\beta F_0 + (cE_0+cP_0)\right]\frac{\partial \beta}{c\partial t} \\
&+\gamma^3\left[2F_0 + \beta(cE_0+cP_0)\right]\frac{\partial \beta}{\partial z} \\
&= -\rho\left(\kappa_0+\sigma_0\right) F_0,
\end{aligned} \tag{23.16}$$

where E_0, F_0, and P_0 are the radiation energy density, the radiative flux, and the radiation pressure in the comoving frame, respectively.

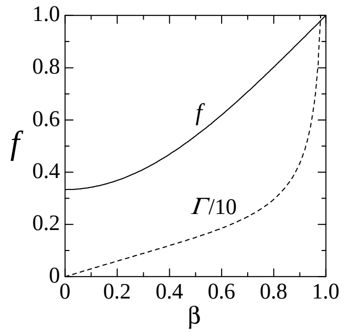

図 23.1: Eddington factor f and the factor Γ as a function of the flow speed β.

In the present vertically one-dimensional flow, the closure relation (23.7) becomes

$$P_0 = f(\tau, \beta) E_0, \tag{23.17}$$

where $f(\tau, \beta)$ is the Eddington factor, and generally a function of the optical depth, the flow speed, and the velocity gradient (Fukue 2008b, d). In Fukue (2008c) the Eddington factor is set to be 1/3, while in this paper we adopt the following form:

$$f(\beta) = \frac{1+3\beta^2}{3+\beta^2}, \tag{23.18}$$

which is 1/3 for $\beta = 0$ and approaches unity as $\beta \to 1$ (Fukue 2009). The behavior of this Eddington factor is shown in figure 1.

23.1.3 Steady Plane-Parallel Flow

Let us further suppose a time-independent steady flow in the vertical direction. In this case the transfer equation and moment equations in the comoving frame become

$$\gamma(\mu_0+\beta)\frac{dI_0}{dz} - \gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{d\beta}{dz}$$
$$= \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi}\right], \tag{23.19}$$

$$\gamma\frac{dF_0}{dz} + \gamma\beta\frac{dcE_0}{dz} + \gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{d\beta}{dz}$$
$$= \rho\left(j_0 - \kappa_0 cE_0\right), \tag{23.20}$$

$$\gamma\frac{dcP_0}{dz} + \gamma\beta\frac{dF_0}{dz} + \gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{d\beta}{dz}$$
$$= -\rho\left(\kappa_0+\sigma_0\right) F_0. \tag{23.21}$$

Introducing the optical depth defined by

$$d\tau \equiv -\left(\kappa_0+\sigma_0\right)\rho dz, \tag{23.22}$$

and the scattering albedo,

$$a \equiv \frac{\sigma_0}{\kappa_0+\sigma_0}, \tag{23.23}$$

the transfer equation (23.19) and the moment equations (23.20) and (23.21) are finally expressed as

$$\gamma(\mu_0+\beta)\frac{dI_0}{d\tau} - \gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{d\beta}{d\tau}$$
$$= I_0 - \frac{1}{4\pi}\frac{j_0}{\kappa_0+\sigma_0} - a\frac{cE_0}{4\pi}, \tag{23.24}$$

$$\gamma\frac{dF_0}{d\tau} + \gamma\beta\frac{dcE_0}{d\tau} + \gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{d\beta}{d\tau}$$
$$= -\frac{j_0}{\kappa_0+\sigma_0} + (1-a)cE_0, \tag{23.25}$$

$$\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} + \gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{d\beta}{d\tau}$$
$$= F_0. \tag{23.26}$$

When the flow speed is spatially constant, as we assume in what follows, these equations become

$$\gamma(\mu_0+\beta)\frac{dI_0}{d\tau} = I_0 - \frac{1}{4\pi}\frac{j_0}{\kappa_0+\sigma_0} - a\frac{cE_0}{4\pi}, \tag{23.27}$$

$$\gamma\frac{dF_0}{d\tau} + \gamma\beta\frac{dcE_0}{d\tau} = -\frac{j_0}{\kappa_0+\sigma_0} + (1-a)cE_0, \tag{23.28}$$

$$\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} = F_0. \tag{23.29}$$

In this paper, we consider this uniform flow case, and examine the relativistic moment equations (23.28) and (23.29) under the Eddington approximation with a variable Eddington factor for several situations.

In equation (23.27), the term $(\mu_0 + \beta)$ on the left-hand side comes from the aberration effect (23.13). We emphasize that this aberration term plays the very important role in the relativistic radiative flow (cf. Fukue 2010b).

23.2 Radiative Equilibrium

We first consider the case of the radiative equilibrium (RE); RE without heating and cooling (Fukue 2008c), RE with heating, and RE with advective cooling. As already stated, the flow speed β is assumed to be constant for simplicity. We further assume that the Eddington factor depends only on the flow speed, and therefore, is constant. Under these situations, we seek analytical solutions of the relativistic moment equations (23.28) and (23.29).

23.2.1 Radiative Equilibrium without Heating and Cooling

If the radiative equilibrium holds in the whole flow, and there is no heating or cooling sources in the flow, then $j_0 = \kappa_0 c E_0$, and the relativistic moment equations (23.28) and (23.29) become

$$\gamma \frac{dF_0}{d\tau} + \gamma\beta \frac{dcE_0}{d\tau} = 0, \tag{23.30}$$

$$\gamma \frac{dcP_0}{d\tau} + \gamma\beta \frac{dF_0}{d\tau} = F_0. \tag{23.31}$$

With the help of the closure relation (23.17), we can eliminate E_0 and P_0 to yield

$$\frac{dF_0}{d\tau} = -\Gamma F_0, \tag{23.32}$$

where

$$\Gamma \equiv \frac{\beta}{\gamma(f-\beta^2)} \tag{23.33}$$

is a function of the flow speed, and shown in figure 1. This equation (23.32) is easily integrated to give the solution for the radiative flux F_0 in the comoving frame:

$$F_0 = F_s e^{-\Gamma\tau}, \tag{23.34}$$

where F_s is the radiative flux at $\tau = 0$. Similarly, the radiation energy density E_0 and the radiation pressure P_0 in the comoving frame are also analytically obtained as

$$cP_0 = fcE_0 = cP_s + F_s \frac{f}{\beta}\left(1 - e^{-\Gamma\tau}\right), \tag{23.35}$$

where P_s is also the value at $\tau = 0$.

Here, we impose the boundary condition at $\tau = 0$:

$$cP_s = 2fF_s \quad \text{at} \quad \tau = 0. \tag{23.36}$$

At last, the analytical solutions become

$$F_0 = F_s e^{-\Gamma\tau}, \tag{23.37}$$

$$cP_0 = fcE_0 = F_s \frac{f}{\beta}\left(1 + 2\beta - e^{-\Gamma\tau}\right). \tag{23.38}$$

These analytical solutions are *just* those found in Fukue (2008c), where the radiative transfer equations in the inertial frame were solved to give the analytical solutions in the inertial frame, and the solutions in the comoving were transformed from them.

For the convenience of readers, the analytical solutions, which are essentially same as those in Fukue (2008c), are shown in figure 2 as a function of the optical depth for several values of the flow speed. The values of β are 0 to 0.9 in steps of 0.1. In figure 2, the solid curves represent the present solutions, whereas the dashed ones mean those in Fukue (2008c), where f is fixed as $f = 1/3$.

As already stated in Fukue (2008c), except for the case of $\beta = 0$, the radiative flux in the comoving frame exponentially decreases with the optical depth, while both the radiation energy density and the radiation pressure approach the constant value at the large optical depth. In the non-relativistic limit of $\beta \to 0$, the solutions reduce to the usual Milne-Eddington ones; $F_0 = F_s$ and $cP_0 = F_s(2f + \tau)$.

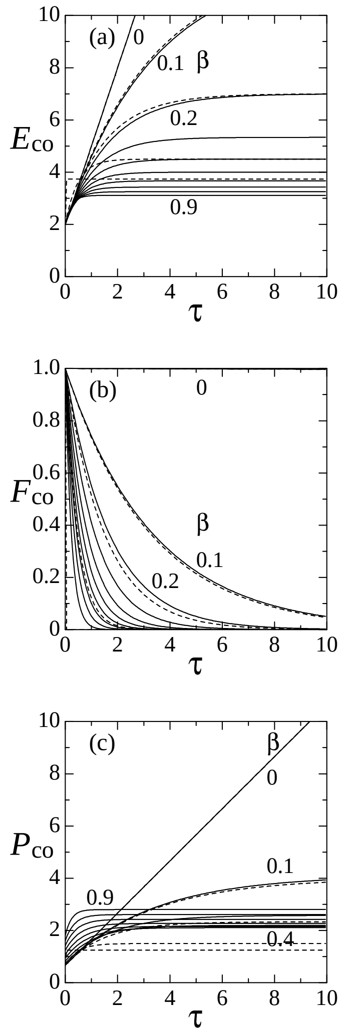

図 23.2: Comoving solutions for relativistic plane-parallel flows in the RE case without heating and cooling: (a) Normalized radiation energy density, (b) normalized radiative flux, and (c) normalized radiation pressure. The values of β are 0 to 0.9 in steps of 0.1. The solid curves represent the present solutions, whereas the dashed ones mean those in Fukue (2008c). In the non-relativistic limit of $\beta \to 0$, the solutions reduce to the usual Milne-Eddington ones; $F_0 = F_{\rm s}$ and $cP_0 = F_{\rm s}(2f + \tau)$.

Analytical solutions for the specific intensity are also obtained, although we skip it since we concentrate the analytical solutions of the radiative moments in this paper.

23.2.2 Radiative Equilibrium with Internal Heating

If there is the internal heating, but the gas pressure is ignored, then the radiative equilibrium condition is modified as

$$j_0 = \frac{q^+}{\rho} + \kappa_0 c E_0, \tag{23.39}$$

where q^+ is the heating rate per unit volume, and the relativistic moment equations (23.28) and (23.29) become

$$\gamma \frac{dF_0}{d\tau} + \gamma\beta \frac{dcE_0}{d\tau} = -\frac{q^+}{\rho(\kappa_0 + \sigma_0)} \equiv -q^*, \tag{23.40}$$

$$\gamma \frac{dcP_0}{d\tau} + \gamma\beta \frac{dF_0}{d\tau} = F_0. \tag{23.41}$$

For simplicity, we assume the mass heating rate q^* is constant. Then, with the help of the closure relation (23.17), we can eliminate E_0 and P_0 to yield

$$\frac{dF_0}{d\tau} = -\Gamma F_0 - \frac{f}{\beta}\Gamma q^*. \tag{23.42}$$

This equation (23.42) is also integrated to give the solution for the radiative flux F_0 in the comoving frame:

$$F_0 = F_1 e^{-\Gamma\tau} - q^* \frac{f}{\beta}, \tag{23.43}$$

where F_1 is now an integration constant.

This solution (23.43) becomes zero at some optical depth. Hence, we impose the boundary condition at the flow base $\tau_{\rm b}$:

$$F_0 = 0 \quad \text{at} \quad \tau = \tau_{\rm b}. \tag{23.44}$$

Under this boundary condition, the solution (23.43) is rewritten as

$$F_0 = q^* \frac{f}{\beta} \left[e^{\Gamma(\tau_{\rm b} - \tau)} - 1 \right]. \tag{23.45}$$

Inserting this solution (23.45) into equation (23.41), the radiation pressure P_0 in the comoving frame is also analytically integrated to give

$$cP_0 = fcE_0 = cP_1 + q^* \frac{f^2}{\beta^2} \left[1 - e^{\Gamma(\tau_b - \tau)}\right] + q^* \frac{f}{\gamma\beta}(\tau_b - \tau), \tag{23.46}$$

where P_1 is also an integration constant.

Here, we impose the boundary condition at $\tau = 0$:

$$cP_s = 2fF_s, \tag{23.47}$$

where

$$F_s = q^* \frac{f}{\beta}\left(e^{\Gamma\tau_b} - 1\right), \tag{23.48}$$

$$cP_s = cP_1 + q^* \frac{f^2}{\beta^2}\left(1 - e^{\Gamma\tau_b}\right) + q^* \frac{f}{\gamma\beta}\tau_b. \tag{23.49}$$

Hence, the integration constant cP_1 is expressed as

$$cP_1 = q^* \frac{f^2}{\beta^2}(1 + 2\beta)\left(e^{\Gamma\tau_b} - 1\right) - q^* \frac{f}{\gamma\beta}\tau_b. \tag{23.50}$$

At last, the analytical solutions (23.45) and (23.46) are expressed as

$$\frac{F_0}{F_s} = \frac{e^{\Gamma(\tau_b - \tau)} - 1}{e^{\Gamma\tau_b} - 1}, \tag{23.51}$$

$$\frac{cP_0}{F_s} = \frac{fcE_0}{F_s} = \frac{f}{\beta}\frac{(1 + 2\beta)e^{\Gamma\tau_b} - 2\beta - e^{\Gamma(\tau_b - \tau)} - \dfrac{\beta}{\gamma f}\tau}{e^{\Gamma\tau_b} - 1}. \tag{23.52}$$

These analytical solutions are shown in figure 3 as a function of the optical depth for several values of the flow speed. The values of β are 0 to 0.9 in steps of 0.1. As is seen in figure 3, the overall behavior of solutions in the RE case with heating is similar to that in the RE case without heating. However, in this case the radiative flux F_0 in the comoving frame becomes zero at a finite optical depth τ_b, which is set to be 10 in the present case.

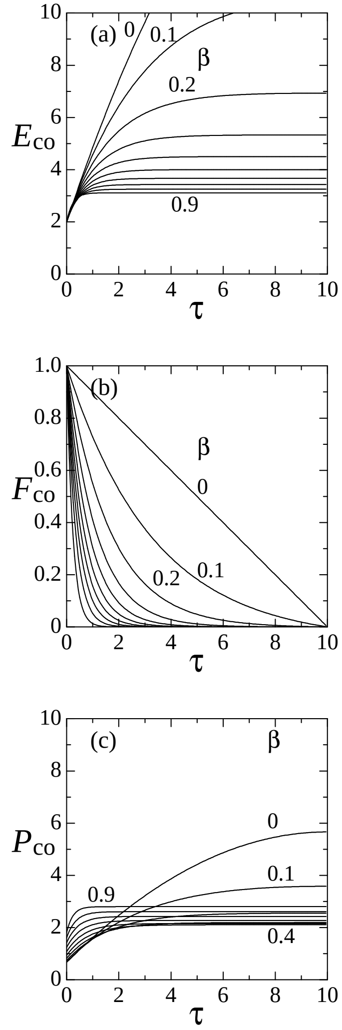

図 23.3: Comoving solutions for relativistic plane-parallel flows in the RE case with heating: (a) Normalized radiation energy density, (b) normalized radiative flux, and (c) normalized radiation pressure. The values of β are 0 to 0.9 in steps of 0.1. The optical depth τ_b at the flow base is set to be 10.

In the non-relativistic limit of $\beta \to 0$, these analytical solutions (23.51) and (23.52) reduce to

$$F_0 = F_\mathrm{s}\left(1 - \frac{\tau}{\tau_\mathrm{b}}\right), \tag{23.53}$$

$$cP_0 = F_\mathrm{s}\left(\frac{2}{3} + \tau - \frac{\tau^2}{2\tau_\mathrm{b}}\right). \tag{23.54}$$

These analytical solutions are *just* those found in Fukue and Akizuki (2006), where the non-relativistic radiative transfer equations of the plane-parallel disk with a finite optical depth were analytically solved for several situations.

23.3 Local Thermodynamic Equilibrium

Next, we consider the case of the local thermodynamic equilibrium (LTE); LTE with constant temperature and LTE with temperature gradient. If the local thermodynamic equilibrium (LTE) holds in the comoving frame,

$$\frac{j_0}{4\pi} = \kappa_0 B_0, \tag{23.55}$$

where B_0 $(= \sigma T_0^4/\pi)$ is the frequency-integrated blackbody intensity in the comoving frame, T_0 being the blackbody temperature, and generally a function of the height z or the optical depth τ.

In this case, the relativistic moment equations (23.28) and (23.29) become

$$\gamma\frac{dF_0}{d\tau} + \gamma\beta\frac{dcE_0}{d\tau} = -(1-a)4\pi B_0 + (1-a)cE_0, \tag{23.56}$$

$$\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} = F_0. \tag{23.57}$$

These equations reduce to those of RE when $a = 1$, as in the non-relativistic case.

With the help of the closure relation (23.17), we can eliminate E_0, and we have

$$\gamma\frac{dF_0}{d\tau} + \frac{\gamma\beta}{f}\frac{dcP_0}{d\tau} = (1-a)\frac{cP_0}{f} - (1-a)4\pi B_0, \tag{23.58}$$

$$\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} = F_0. \tag{23.59}$$

Furthermore, after some manipulations we have

$$\frac{d^2F_0}{d\tau^2} + (2-a)\Gamma\frac{dF_0}{d\tau} - \frac{1-a}{\gamma\beta}\Gamma F_0 = -(1-a)\frac{f\Gamma}{\beta}4\pi\frac{dB_0}{d\tau}. \tag{23.60}$$

23.3.1 Uniform Blackbody Case

We first assume that the blackbody intensity is uniform in the whole flow. Then equation (23.72) becomes a homegeneous linear differential one:

$$\frac{d^2F_0}{d\tau^2} + (2-a)\Gamma\frac{dF_0}{d\tau} - \frac{1-a}{\gamma\beta}\Gamma F_0 = 0. \tag{23.61}$$

The general solution of this equation (23.73) is

$$F_0 = C_1(\beta)e^{\lambda_1\tau} + C_2(\beta)e^{\lambda_2\tau}. \tag{23.62}$$

Inserting this solution (23.74) into equation (23.71), we can obtain the general solution for P_0,

$$cP_0 = C_3(\beta) + \left(\frac{1}{\gamma\lambda_1} - \beta\right)C_1e^{\lambda_1\tau} + \left(\frac{1}{\gamma\lambda_2} - \beta\right)C_2e^{\lambda_2\tau}. \tag{23.63}$$

Here, the coefficients C_i's are functions of β in the present relativistic case. In addition, the indeces λ_i's are expressed as

$$\lambda_{1,2} = -\frac{1}{2}\left[(2-a)\Gamma \pm \sqrt{(2-a)^2\Gamma^2 + \frac{4(1-a)}{\gamma\beta}\Gamma}\right], \tag{23.64}$$

and shown in figure 5 as a function of β for several values of a. The values of a are 0 to 1 in steps of 0.1. The index λ_1 reduces to $-\Gamma$ when $a = 1$.

As is well-known, the solutions of the inhomogenious equation are the combination of the general solutions of the homogenious part and the special solution of the inhomogenious part. When B_0 is constant, after some manipulations, the

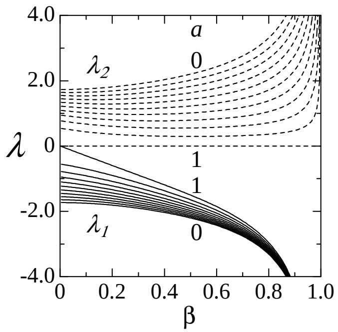

図 23.4: Indeces λ_1 and λ_2 as a function of β for several values of a.

general solutions for equations (23.70) and (23.71) with the uniform blackbody case become as

$$F_0 = C_1 e^{\lambda_1 \tau} + C_2 e^{\lambda_2 \tau}, \tag{23.65}$$

$$\begin{aligned} cP_0 &= 4f\pi B_0 \\ &\quad + \left(\frac{1}{\gamma\lambda_1} - \beta\right) C_1 e^{\lambda_1 \tau} + \left(\frac{1}{\gamma\lambda_2} - \beta\right) C_2 e^{\lambda_2 \tau}, \end{aligned} \tag{23.66}$$

where the coefficients C_1 and C_2 are now some constants and to be determined by the boundary conditions.

We now impose the boundary conditions at the flow top of $\tau = 0$ and the flow base of $\tau = \tau_b$. At the flow top we set

$$F_0 = F_s \quad \text{at} \quad \tau = 0, \tag{23.67}$$

while at the flow base we set

$$F_0 = 0 \quad \text{at} \quad \tau = \tau_b. \tag{23.68}$$

In this case, the coefficients become

$$C_1 = F_s \frac{e^{(\lambda_2 - \lambda_1)\tau_b}}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1}, \tag{23.69}$$

$$C_2 = -F_s \frac{1}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1}. \tag{23.70}$$

Hence, we finally obtain the solutions for equations (23.70) and (23.71) in the uniform blackbody case as

$$\begin{aligned} \frac{F_0}{F_s} &= \frac{e^{(\lambda_2 - \lambda_1)\tau_b}}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1} e^{\lambda_1 \tau} \\ &\quad - \frac{1}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1} e^{\lambda_2 \tau}, \end{aligned} \tag{23.71}$$

$$\begin{aligned} \frac{cP_0}{F_s} &= \frac{fcE_0}{F_s} \\ &= \frac{e^{(\lambda_2 - \lambda_1)\tau_b}}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1} \left(\frac{1}{\gamma\lambda_1} - \beta\right) e^{\lambda_1 \tau} \\ &\quad - \frac{1}{e^{(\lambda_2 - \lambda_1)\tau_b} - 1} \left(\frac{1}{\gamma\lambda_2} - \beta\right) e^{\lambda_2 \tau} \\ &\quad + 4f \frac{\pi B_0}{F_s}. \end{aligned} \tag{23.72}$$

Examples of these analytical solutions (23.83) and (23.84) are shown in figure 6 as a function of the optical depth for several values of the flow speed. The values of β are 0.1 to 0.9 in steps of 0.2. The other parameters are set as $\pi B_0/F_s = 1$ and $a = 0.5$.

As is seen in figure 6, the qualitative behavior of LTE solutions is similar to that of RE solutions; the radiative flux in the comoving frame exponetially decreases with the optical depth, while both the radiation energy density and the radiation pressure approach the constant value at the large optical depth. Furthermore, at the sufficiently large optical depth, irrespective of the value of β, the radiation energy density approach the same value of LTE; $cE_0 \to 4\pi B_0$. This is the characteristics of the LTE case.

It should be noted that in the non-relativistic limit of $\beta \to 0$ the solutions (23.83) and (23.84) reduce to the relevant ones for the Earth's atmosphere (cf. Thomas, Stamnes 1999). In other words, the present solution is the relativistic generalization of the previously known solutions.

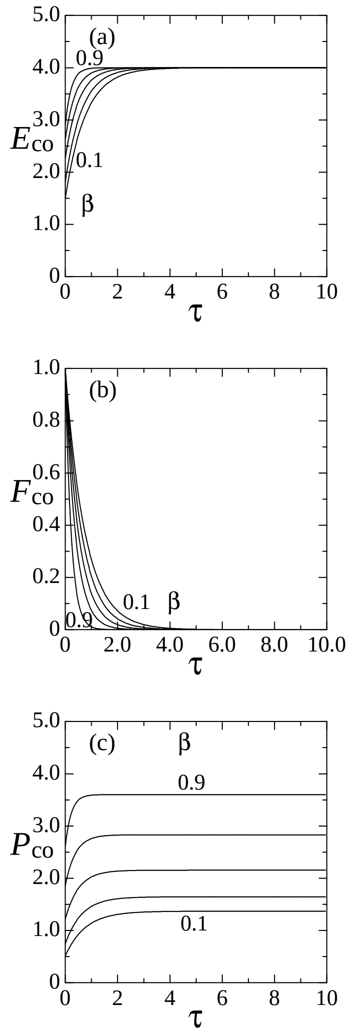

図 23.5: Comoving solutions for relativistic plane-parallel flows in the LTE case with uniform blackbody intensity: (a) Normalized radiation energy density, (b) normalized radiative flux, and (c) normalized radiation pressure. The values of β are 0.1 to 0.9 in steps of 0.2. The other parameters are set as $\pi B_0/F_\mathrm{s} = 1$ and $a = 0.5$.

23.3.2 Eddington-Barbier Case

We also examine the non-uniform case. In particular, we assume the linear approximation for B_0;

$$B_0(\tau) = B_{00} + B_1\tau \tag{23.73}$$

in the comoving frame. This is so-called the Eddington-Barbier relation in the static atmosphere.

Using similar procedure and after some manipulations, the general solutions for equations (23.70) and (23.71) with the Eddington-Barbier relation become as

$$F_0 = C_1 e^{\lambda_1\tau} + C_2 e^{\lambda_2\tau} + 4\gamma f\pi B_1, \tag{23.74}$$

$$\begin{aligned} cP_0 = & \left(\frac{1}{\gamma\lambda_1} - \beta\right) C_1 e^{\lambda_1\tau} + \left(\frac{1}{\gamma\lambda_2} - \beta\right) C_2 e^{\lambda_2\tau} \\ & + 4f\pi B_0(\tau) + \frac{\gamma\beta f}{1-a} 4\pi B_1, \end{aligned} \tag{23.75}$$

where the coefficients C_1 and C_2 are constants to be determined by the boundary conditions, and the indeces λ_1 and λ_2 are given in equation (23.76).

We now impose the boundary conditions at the flow top of $\tau = 0$ and the infinite optical depth instead of the finite optical depth for simplicity; we here suppose a semi-infinte plane-parallel atmosphere. In order for the solution not to diverge at the infinite optical depth of $\tau \to \infty$, the coefficient $C_2 = 0$ since $\lambda < 0$ and $\lambda \geq 0$. At the flow top we set

$$F_0 = F_\mathrm{s} \quad \text{at} \quad \tau = 0. \tag{23.76}$$

Hence, the coefficient C_1 is determined as

$$C_1 = F_\mathrm{s} - 4\gamma f\pi B_1. \tag{23.77}$$

Since there are two freedom, B_{00} and B_1 in this case, we set one additional boundary condition at the flow top,

$$cP_0 = 2fF_\mathrm{s} \quad \text{at} \quad \tau = 0. \tag{23.78}$$

Then, the additional condition is obtained,

$$b_0 = \frac{1}{2} - \frac{1}{4f}\left(\frac{1}{\gamma\lambda_1} - \beta\right) + \gamma\left(\frac{1}{\gamma\lambda_1} - \frac{2-a}{1-a}\beta\right) b_1, \tag{23.79}$$

where

$$b_0 = \frac{\pi B_{00}}{F_s}, \tag{23.80}$$

$$b_1 = \frac{\pi B_1}{F_s}. \tag{23.81}$$

Hence, we finally obtain the solutions for equations (23.70) and (23.71) in the Eddington-Barbier case as

$$\frac{F_0}{F_s} = 4\gamma f b_1 + (1 - 4\gamma f b_1)\, e^{\lambda_1 \tau}, \tag{23.82}$$

$$\begin{aligned} \frac{cP_0}{F_s} &= \frac{fcE_0}{F_s} \\ &= 2f - \left(\frac{1}{\gamma\lambda_1} - \beta\right)(1 - 4\gamma f b_1)\left(1 - e^{\lambda_1\tau}\right) \\ &\quad +4fb_1\tau, \end{aligned} \tag{23.83}$$

where

$$\lambda_1 = -\frac{1}{2}\left[(2-a)\Gamma + \sqrt{(2-a)^2\Gamma^2 + \frac{4(1-a)}{\gamma\beta}\Gamma}\right], \tag{23.84}$$

and

$$4\gamma f b_1 = \frac{2f - \left(\frac{1}{\gamma\lambda_1} - \beta\right) - 4fb_0}{-\frac{1}{\gamma\lambda_1} + \frac{2-a}{1-a}\beta}. \tag{23.85}$$

The parameters are a, b_0, and β. In addition, there are two conditions for the solution to be physical;

$$b_0 > 0, \tag{23.86}$$

$$4\gamma f b_1 < 1. \tag{23.87}$$

These solutions (23.94) and (23.95) are the *relativistic Eddingto-Barbier solutions.*

Examples of these analytical solutions (23.94) and (23.95) are shown in figure 7 as a function of the optical depth for several values of the flow speed. The values of β are 0.1 to 0.4 in steps of 0.1. The other parameters are set as $\pi B_0/F_s = 1$ and $a = 0.9$. In figure 7 the solid curves represent the present solutions, whereas the dashed ones mean those of the RE case without heating and cooling. In addition, the blackbody functions B_0 in the present case are also shown in figure 8. Due to the conditions (23.98) and (23.99), there is no physical solutions when the flow speed becomes large.

As is seen in figure 7, the behavior of LTE solutions of the non-uniform blackbody case is rather different from that of LTE solutions of the uniform case; the radiative flux in the comoving frame exponetially decreases with the optical depth but approaches some finite values, while both the radiation energy density and the radiation pressure increases linearly at the large optical depth. Indeed, at large optical depth analytical solutions (23.94) and (23.95) become, respectively,

$$\frac{F_0}{F_s} = 4\gamma f b_1, \tag{23.88}$$

$$\frac{cP_0}{F_s} = \frac{fcE_0}{F_s} = 4f\left(\frac{\gamma\beta}{1-a}b_1 + b_0 + b_1\tau\right). \tag{23.89}$$

Hence, at the large optical depth the radiation energy density approach the same value of LTE; $cE_0 \rightarrow 4\pi B_0$. This is the characteristics of the LTE case.

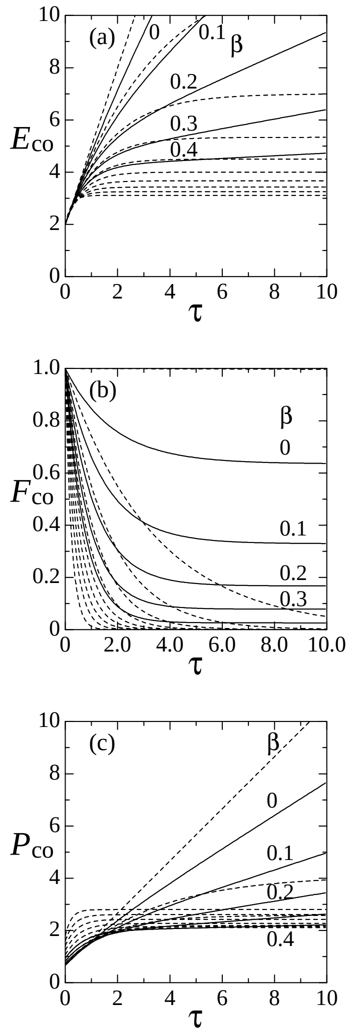

図 23.6: Comoving solutions for relativistic plane-parallel flows in the LTE case with the Eddington-Barbier relation: (a) Normalized radiation energy density, (b) normalized radiative flux, and (c) normalized radiation pressure. The values of β are 0.1 to 0.4 in steps of 0.1. The other parameters are set as $\pi B_0/F_\mathrm{s} = 1$ and $a = 0.9$. The solid curves represent the present solutions, whereas the dashed ones mean those of the RE case without heating and cooling.

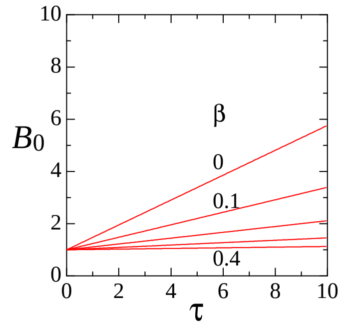

図 23.7: Blackbody function in the LTE case with the Eddington-Barbier relation. The values of β are 0.1 to 0.4 in steps of 0.1. The other parameters are set as $\pi B_0/F_\mathrm{s} = 1$ and $a = 0.9$.