# 第32章 Relativistic Radiative Transfer Spherical Comoving Frame variables $I_0 E_0 F_0 P_0$ in physics

After Fukue, J. 2011, AinA,

## 32.1 Relativistic Radiative Transfer Equation

The radiative transfer equations are given in several literatures (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 1998, 2008; Mihalas, Auer 2001; Peraiah 2002; Castor 2004). The basic equations for relativistic radiation hydrodynamics are given in, e.g., the appendix E of Kato et al. (2008) in general and vertical forms (see also Fukue 2008c, 2010c).

### 32.1.1 General Form

In a general form the radiative transfer equation in the *mixed frame*, where the variables in the inertial and comoving frames are used, is expressed as

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla}) I = \left(\frac{\nu}{\nu_0}\right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi}\right], \quad (32.1)$$

where $c$ is the speed of light. In the left-hand side the frequency-integrated specific intensity $I$ and the direction cosine $\boldsymbol{l}$ are quantities measured in the inertial (fixed) frame. In the right-hand side, on the other hand, the mass density $\rho$, the frequency-integrated mass emissivity $j_0$, the frequency-integrated mass absorption coefficient $\kappa_0$, the frequency-integrated mass scattering coefficient $\sigma_0$, the frequency-integrated specific intensity $I_0$, and the frequency-integrated radiation energy density $E_0$ are quantities measured in the comoving (fluid) frame. In this paper, instead of the weakly anisotropic Thomson scattering, we assume that the scattering is isotropic for simplicity.

The Doppler effect, the aberration, and the transformation of the intensities are expressed as

$$\frac{\nu}{\nu_0} = \gamma(1+\boldsymbol{\beta}\cdot\boldsymbol{l}_0), \quad (32.2)$$

$$\boldsymbol{l} = \frac{\nu_0}{\nu}\left[\boldsymbol{l}_0 + \left(\frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\cdot\boldsymbol{l}_0 + \gamma\right)\boldsymbol{\beta}\right], \quad (32.3)$$

$$I = \left(\frac{\nu}{\nu_0}\right)^4 I_0, \quad (32.4)$$

where $\nu$ and $\nu_0$ are the frequencies measured in the inertial and comoving frames, respectively, the direction cosine $\boldsymbol{l}_0$ measureed in the comoving frame, $\boldsymbol{\beta}$ ($=\boldsymbol{v}/c$) the normalized velocity, $\boldsymbol{v}$ being the flow velocity, and $\gamma$ ($=1/\sqrt{1-\beta^2}$) the Lorentz factor, $\beta$ being $v/c$.

The zeroth and first moment equations are, respectively,

$$\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} = \rho\gamma(j_0 - \kappa_0 cE_0) - \rho\gamma(\kappa_0+\sigma_0)\boldsymbol{\beta}\cdot\boldsymbol{F}_0, \quad (32.5)$$

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} = \rho\gamma\frac{\beta^i}{c}(j_0-\kappa_0 cE_0) - \rho(\kappa_0+\sigma_0)\frac{\gamma-1}{\beta^2}\frac{\beta^i}{c}(\boldsymbol{\beta}\cdot\boldsymbol{F}_0) - \frac{1}{c}\rho(\kappa_0+\sigma_0)F_0^i, \quad (32.6)$$

where the frequency-integrated radiation energy density $E$, the frequency-integrated radiative flux $\boldsymbol{F}$, and the frequency-integrated radiation stress $P^{ik}$ are measured in the inertial frame, while those with the subscript 0 are measured in the comoving frame.

As a closure relation, we adopt the Eddington approximation *in the comoving frame*:

$$P_0^{ik} = f^{ik} E_0, \tag{32.7}$$

where $f^{ik}$ is the Eddington tensor, which is generally a function of the optical depth and flow speed in the relativistic radiative flow.

### 32.1.2 Spherical Expression in the Comoving Frame

Let us suppose a relativistic spherical flow, e.g., a luminous black hole wind. In the spherical geometry with the radius $r$ the transfer equation (32.1) is expressed as

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = \left(\frac{\nu}{\nu_0}\right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0)\, I_0 + \sigma_0 \frac{cE_0}{4\pi}\right] \tag{32.8}$$

where $\mu$ is the direction cosine in the inertial frame. Inserting the transformation (32.4) in the left-hand side, this equation (32.8) becomes

$$\frac{\nu}{\nu_0}\left(\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_0}{\partial \mu}\right) - 4\frac{\nu}{\nu_0^2} I_0 \left(\frac{1}{c}\frac{\partial \nu_0}{\partial t} + \mu\frac{\partial \nu_0}{\partial z} + \frac{1-\mu^2}{r}\frac{\partial \nu_0}{\partial \mu}\right)$$
$$= \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0)\, I_0 + \sigma_0 \frac{cE_0}{4\pi}\right]. \tag{32.9}$$

To calculate the derivatives of $I_0$ (Mihalas, Mihalas 1984), we apply the chain rules and after some manipulations we have

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{r\mu\nu} &= \left.\frac{\partial}{\partial t}\right|_{r\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial t}\right|_{r\mu_0\nu_0}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial t}\right|_{r\mu_0\nu_0}\frac{\partial}{\partial \nu_0} \\
&= \left.\frac{\partial}{\partial t}\right|_{r\mu_0\nu_0} - \gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial t}\frac{\partial}{\partial \mu_0} - \gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial t}\frac{\partial}{\partial \nu_0}, \qquad (32.10) \\
\left.\frac{\partial}{\partial r}\right|_{t\mu\nu} &= \left.\frac{\partial}{\partial r}\right|_{t\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial r}\right|_{t\mu_0\nu_0}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial r}\right|_{t\mu_0\nu_0}\frac{\partial}{\partial \nu_0} \\
&= \left.\frac{\partial}{\partial r}\right|_{t\mu_0\nu_0} - \gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial r}\frac{\partial}{\partial \mu_0} - \gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial r}\frac{\partial}{\partial \nu_0}, \qquad (32.11) \\
\left.\frac{\partial}{\partial \mu}\right|_{rt\nu} &= \left.\frac{\partial \mu_0}{\partial \mu}\right|_{rt\nu_0}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial \mu_0}\right|_{rt\nu_0}\frac{\partial}{\partial \nu_0} \\
&= \gamma^2(1+\beta\mu_0)^2\frac{\partial}{\partial \mu_0} - \gamma^2\beta(1+\beta\mu_0)\nu_0\frac{\partial}{\partial \nu_0}, \qquad (32.12)
\end{aligned}$$

where $\mu_0$ is the direction cosine in the comoving frame. In addition, the Doppler shift (32.2) and the aberration (32.3) are respectively expressed as

$$\frac{\nu}{\nu_0} = \gamma(1+\beta\mu_0), \tag{32.13}$$
$$\mu = \frac{\mu_0+\beta}{1+\beta\mu_0}. \tag{32.14}$$

Using these expressions, after some manipulations we have the radiative transfer equation in the comoving frame for the spherical flow:

$$\begin{aligned}
&\gamma(1+\beta\mu_0)\frac{1}{c}\frac{\partial I_0}{\partial t} + \gamma(\mu_0+\beta)\frac{\partial I_0}{\partial r} + \gamma(1+\beta\mu_0)\frac{1-\mu_0}{r}\frac{\partial I_0}{\partial \mu_0} + 4\gamma\beta\frac{1-\mu_0^2}{r} I_0 \\
&-\gamma^3(1+\beta\mu_0)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{1}{c}\frac{\partial \beta}{\partial t} \\
&-\gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{\partial \beta}{\partial r} \\
&= \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0\frac{cE_0}{4\pi}\right]. \qquad (32.15)
\end{aligned}$$

Integrating the transfer equation (32.15) over a solid angle, we have the zeroth and first moment equations in the comoving frame for the spherical flow:

$$\begin{aligned}
&\gamma\frac{\partial cE_0}{c\partial t} + \gamma\frac{\partial F_0}{\partial r} + \gamma\beta\frac{\partial F_0}{c\partial t} + \gamma\beta\frac{\partial cE_0}{\partial r} + \frac{\gamma}{r}\left[2F_0 + \beta(3cE_0 - cP_0)\right] \\
&\quad +\gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{\partial \beta}{c\partial t} + \gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{\partial \beta}{\partial r} \\
&\quad = \rho\left(j_0 - \kappa_0 cE_0\right), \qquad (32.16) \\
&\gamma\frac{\partial F_0}{c\partial t} + \gamma\frac{\partial cP_0}{\partial r} + \gamma\beta\frac{\partial cP_0}{c\partial t} + \gamma\beta\frac{\partial F_0}{\partial r} + \frac{\gamma}{r}\left[2\beta F_0 - cE_0 + 3cP_0\right] \\
&\quad +\gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{\partial \beta}{c\partial t} + \gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{\partial \beta}{\partial r} \\
&\quad = -\rho\left(\kappa_0 + \sigma_0\right) F_0, \qquad (32.17)
\end{aligned}$$

where $E_0$, $F_0$, and $P_0$ are the radiation energy density, the radiative flux, and the radiation pressure in the comoving frame, respectively.

In the present spherical one-dimensional flow, the closure relation (32.7) becomes

$$P_0 = f(\tau, \beta) E_0, \tag{32.18}$$

where $f(\tau, \beta)$ is the variable Eddington factor, and generally a function of the optical depth, the flow speed, and the velocity gradient (Fukue 2008b, d). In the plane-parallel flow (Fukue 2010c), the following form was adopted:

$$f(\beta) = \frac{1+3\beta^2}{3+\beta^2}, \tag{32.19}$$

which is 1/3 for $\beta = 0$ and approaches unity as $\beta \to 1$ (Fukue 2009). In the present spherical case, we adopt alternative appropriate forms, which are shown later.

### 32.1.3 Steady Spherical Flow

Let us further suppose a time-independent steady flow in the radial direction. In this case the transfer equation and moment equations in the comoving frame become

$$\begin{aligned}
&\gamma(\mu_0+\beta)\frac{dI_0}{dr} - \gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{d\beta}{dr} \\
&\quad +\gamma(1+\beta\mu_0)\frac{1-\mu_0^2}{r}\frac{\partial I_0}{\partial \mu_0} + 4\gamma\beta\frac{1-\mu_0^2}{r} I_0 \\
&\quad = \rho\left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi}\right],
\end{aligned} \tag{32.20}$$

$$\begin{aligned}
&\gamma\frac{dF_0}{dr} + \gamma\beta\frac{dcE_0}{dr} + \gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{d\beta}{dr} + \frac{\gamma}{r}\left[2F_0 + \beta(3cE_0 - cP_0)\right] \\
&\quad = \rho\left(j_0 - \kappa_0 cE_0\right),
\end{aligned} \tag{32.21}$$

$$\begin{aligned}
&\gamma\frac{dcP_0}{dr} + \gamma\beta\frac{dF_0}{dr} + \gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{d\beta}{dr} + \frac{\gamma}{r}\left[2\beta F_0 - cE_0 + 3cP_0\right] \\
&\quad = -\rho\left(\kappa_0+\sigma_0\right) F_0.
\end{aligned} \tag{32.22}$$

Introducing the optical depth defined by

$$d\tau \equiv -\left(\kappa_0+\sigma_0\right)\rho dr, \tag{32.23}$$

and the scattering albedo,

$$a \equiv \frac{\sigma_0}{\kappa_0+\sigma_0}, \tag{32.24}$$

the transfer equation (32.20) and the moment equations (32.21) and (32.22) are finally expressed as

$$\begin{aligned}
&\gamma(\mu_0+\beta)\frac{dI_0}{d\tau} - \gamma(1+\beta\mu_0)\frac{1-\mu_0^2}{\rho(\kappa_0+\sigma_0)r}\frac{\partial I_0}{\partial \mu_0} - 4\gamma\beta\frac{1-\mu_0^2}{\rho(\kappa_0+\sigma_0)r} I_0 \\
&\quad -\gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0\right]\frac{d\beta}{d\tau} \\
&\quad = I_0 - \frac{1}{4\pi}\frac{j_0}{\kappa_0+\sigma_0} - a\frac{cE_0}{4\pi},
\end{aligned} \tag{32.25}$$

$$\begin{aligned}
&\gamma\frac{dF_0}{d\tau} + \gamma\beta\frac{dcE_0}{d\tau} - \frac{\gamma}{\rho(\kappa_0+\sigma_0)r}\left[2F_0 + \beta(3cE_0 - cP_0)\right] \\
&\quad +\gamma^3\left[2\beta F_0 + (cE_0 + cP_0)\right]\frac{d\beta}{d\tau} = -\frac{j_0}{\kappa_0+\sigma_0} + (1-a)cE_0,
\end{aligned} \tag{32.26}$$

$$\begin{aligned}
&\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} - \frac{\gamma}{\rho(\kappa_0+\sigma_0)r}\left[2\beta F_0 - cE_0 + 3cP_0\right] \\
&\quad +\gamma^3\left[2F_0 + \beta(cE_0 + cP_0)\right]\frac{d\beta}{d\tau} = F_0.
\end{aligned} \tag{32.27}$$

Here, we further introduce the spherical variables by

$$L_0 \equiv 4\pi r^2 F_0, \tag{32.28}$$

$$D_0 \equiv 4\pi r^2 cE_0, \tag{32.29}$$

$$Q_0 \equiv 4\pi r^2 cP_0, \tag{32.30}$$

and the moment equations (32.26) and (32.27) become

$$\begin{aligned}
&\gamma\frac{dL_0}{d\tau} + \gamma\beta\frac{dD_0}{d\tau} - \gamma\beta\frac{D_0 - Q_0}{\rho(\kappa_0+\sigma_0)r} + \gamma^3\left(2\beta L_0 + D_0 + Q_0\right)\frac{d\beta}{d\tau} \\
&\quad = -\frac{4\pi r^2}{\kappa_0+\sigma_0}\left(j_0 - \kappa_0 cE_0\right),
\end{aligned} \tag{32.31}$$

$$\begin{aligned}
&\gamma\frac{dQ_0}{d\tau} + \gamma\beta\frac{dL_0}{d\tau} - \gamma\frac{Q_0 - D_0}{\rho(\kappa_0+\sigma_0)r} + \gamma^3\left[2L_0 + \beta(D_0 + Q_0)\right]\frac{d\beta}{d\tau} \\
&\quad = L_0,
\end{aligned} \tag{32.32}$$

and the closure relation (32.18) is written as

$$Q_0 = f(\tau, \beta) D_0. \tag{32.33}$$

If we assume the streaming limit of $D_0 = Q_0$ ($f = 1$) with a constant speed in these equations (32.31) and (32.32), we have the exponential type solutions, which were obtained in Fukue (2010b). In this paper we consider more general cases of $Q_0 = fD_0$.

Using this closure relation (32.33), the Eddington factor being not yet determined, and the definition of the optical depth (32.23), the relativistic moment equations (32.31) and (32.32) are expressed as

$$\gamma\frac{dL_0}{d\tau} + \gamma\beta\frac{dD_0}{d\tau} + \gamma\beta\frac{1-f}{r}D_0\frac{dr}{d\tau} + \gamma^3\left[2\beta L_0 + (1+f)D_0\right]\frac{d\beta}{d\tau}$$
$$= -\frac{4\pi r^2}{\kappa_0+\sigma_0}\left(j_0 - \kappa_0 cE_0\right), \tag{32.34}$$
$$\gamma\frac{d(fD_0)}{d\tau} + \gamma\beta\frac{dL_0}{d\tau} - \gamma\frac{1-f}{r}D_0\frac{dr}{d\tau} + \gamma^3\left[2L_0 + \beta(1+f)D_0\right]\frac{d\beta}{d\tau}$$
$$= L_0. \tag{32.35}$$

After several manipulations and rearrangement, the relativistic moment equations (32.34) and (32.35) in the comoving frame are finally expressed as

$$\frac{\gamma(f-\beta^2)}{f}\frac{dL_0}{d\tau} + \gamma\beta\left(\frac{1-f^2}{fr} - \frac{1}{f}\frac{df}{dr}\right)D_0\frac{dr}{d\tau}$$
$$+\gamma^3\left[2\beta\left(1-\frac{1}{f}\right)L_0 + (1+f)\left(1-\frac{\beta^2}{f}\right)D_0\right]\frac{d\beta}{d\tau}$$
$$= -\frac{4\pi r^2}{\kappa_0+\sigma_0}\left(j_0 - \kappa_0 cE_0\right) - \frac{\beta}{f}L_0, \tag{32.36}$$
$$\gamma\left(f-\beta^2\right)\frac{dD_0}{d\tau} + \gamma\left[\frac{df}{dr} - \frac{(1-f)(1+\beta^2)}{r}\right]D_0\frac{dr}{d\tau} + 2\gamma L_0\frac{d\beta}{d\tau}$$
$$= L_0 + \beta\frac{4\pi r^2}{\kappa_0+\sigma_0}\left(j_0 - \kappa_0 cE_0\right). \tag{32.37}$$

After we determine the appropriate form of the variable Eddington factor, we can solve the moment equations (32.36) and (32.37) in some restricted cases.

Before solving the moment equations, we derive a relation between the optical depth $\tau$ and radius $r$. If the flow is steady, as is assumed, the continuity equation for the spherical case is written as

$$4\pi r^2\rho\gamma\beta c = \dot{M}, \tag{32.38}$$

where $\dot{M}$ is the constant mass-outflow rate. Using this continuity equation (32.38), assuming the opacities are constant, and imposing the boundary condition of $\tau = 0$ at $r = \infty$, we can integrate the optical depth (32.23) to give

$$\tau = \frac{\dot{M}(\kappa_0+\sigma_0)}{4\pi\gamma\beta c}\frac{1}{r} = \rho(\kappa_0+\sigma_0)r, \tag{32.39}$$

which is also written as

$$\frac{\tau}{\tau_c} = \frac{r_c}{r}, \tag{32.40}$$

where the subscript c denotes some reference position (core radius). It should be noted that the optical depth at the core radius is related to the core radius by

$$\tau_c = \frac{\dot{m} r_g}{2\gamma\beta r_c}, \tag{32.41}$$

where $\dot{m}$ $(= \dot{M}/\dot{M}_E)$ is the mass-outflow rate normalized by the critical rate $\dot{M}_E$ $(= L_E/c^2)$, $L_E$ being the Eddington luminosity of the central object, and $r_g$ $(= 2GM/c^2)$ is the Schwarzshild radius of the central object. In what follows, we use these relations, if necessary.

## 32.2 Radiative Equilibrium

We first consider the case of the radiative equilibrium (RE) without heating and cooling. If the radiative equilibrium holds in the whole flow, and there is no heating or cooling, then $j_0 = \kappa_0 cE_0$, and the relativistic moment equations (32.36) and (32.37) become

$$\frac{\gamma(f-\beta^2)}{f}\frac{dL_0}{d\tau} + \gamma\beta\left(\frac{1-f^2}{fr} - \frac{1}{f}\frac{df}{dr}\right)D_0\frac{dr}{d\tau}$$
$$+\gamma^3\left[2\beta\left(1-\frac{1}{f}\right)L_0 + (1+f)\left(1-\frac{\beta^2}{f}\right)D_0\right]\frac{d\beta}{d\tau}$$
$$= -\frac{\beta}{f}L_0, \tag{32.42}$$
$$\gamma\left(f-\beta^2\right)\frac{dD_0}{d\tau} + \gamma\left[\frac{df}{dr} - \frac{(1-f)(1+\beta^2)}{r}\right]D_0\frac{dr}{d\tau} + 2\gamma L_0\frac{d\beta}{d\tau}$$
$$= L_0. \tag{32.43}$$

These equations (32.42) and (32.43) are rather complicated yet, since they include the velocity gradient term and the derivative of the radius, whose are connected with the hydrodynamical equations. Of these, except for the central accelerating region, the wind speed weakly depends on the optical depth and is almost constant in the terminal stage. Hence, as already stated, the flow speed $\beta$ is assumed to be constant in this paper. On the other hand, the radius-derivative term depends on the optical depth. Indeed, it is expressed as

$$\frac{dr}{d\tau} = -\frac{1}{(\kappa_0+\sigma_0)\rho} = -\frac{r}{\tau} \propto -\tau^{-2}. \tag{32.44}$$

Instead, the second term on the left-hand side of equation (32.43) can be dropped, if we impose the restricted condition on the Eddington factor as

$$\frac{1-f^2}{r} - \frac{df}{dr} = 0. \tag{32.45}$$

This equation (32.45) is easily integrated to give

$$f = \frac{C(\beta)r^2-1}{C(\beta)r^2+1}, \tag{32.46}$$

where $C(\beta)$ is an integration constant, and generally a function of the constant flow speed $\beta$. We impose the boundary condition at the core radius such as

$$f = \frac{1+3\beta^2}{3+\beta^2} \quad \text{at} \quad r = r_\mathrm{c}, \tag{32.47}$$

and the appropriate Eddington factor requested to the present case finally becomes

$$f = \frac{2\gamma^2(1+\beta^2)\hat{r}^2-1}{2\gamma^2(1+\beta^2)\hat{r}^2+1} = \frac{2\gamma^2(1+\beta^2)-\hat{\tau}^2}{2\gamma^2(1+\beta^2)+\hat{\tau}^2}, \tag{32.48}$$

where $\hat{r} = r/r_\mathrm{c}$ and $\hat{\tau} = \tau/\tau_\mathrm{c}$. This variable Eddington factor (32.48) satisfies the condition: $f \to 1/3$ when $r \to r_\mathrm{c}$ and $\beta \to 0$, and $f \to 1$ when $r \to \infty$ ($\tau \to 0$) or $\beta \to 1$. The behavior of this Eddington factor is shown in figure 1.

Under these restrictive conditions, after several manipulations, equations (32.42) and (32.43) become

$$\frac{dL_0}{d\tau} = -\Gamma L_0, \tag{32.49}$$

$$\gamma\frac{1}{g}\frac{d}{d\tau}\left[g(f-\beta^2)D_0\right] = L_0. \tag{32.50}$$

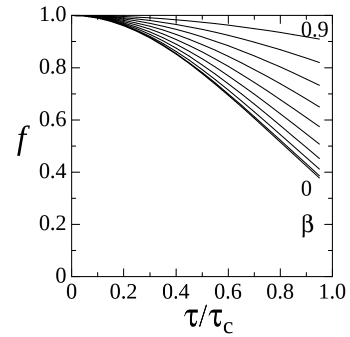


図 32.1: Eddington factor $f$ appropriate for the RE case as a function of the optical depth $\tau$ for the various values of the flow speed $\beta$. The values of $\beta$ are 0 to 0.9 in steps of 0.1 from bottom to top.

In these equations,

$$\Gamma \equiv \frac{\beta}{\gamma(f-\beta^2)} \tag{32.51}$$

is a function of the flow speed and the optical depth and it becomes

$$\Gamma = \frac{\gamma\beta}{1+\beta^2}\frac{2\hat{\tau}^2}{2-\hat{\tau}^2} + \gamma\beta \tag{32.52}$$

for the Eddington factor (32.48), while $g$ is the curvature factor defined by

$$\ln g \equiv -\int_{\tau_\mathrm{c}}^{\tau} \frac{(1-f)(1+\beta^2)}{(f-\beta^2)r}\frac{dr}{d\tau}d\tau', \tag{32.53}$$

and become in the present case

$$g = \frac{\hat{\tau}^3}{2-\hat{\tau}^2}. \tag{32.54}$$

Since the index $\Gamma$ is analyticall expressed by the optical depth, the differential equation (32.49) can analytically integrated to give the comoving luminosity $L_0$. Imposing the boundary condition of $L_\mathrm{s}$ at $\tau = 0$, we finally have the comoving luminosity for the RE case:

$$\frac{L_0}{L_\mathrm{s}} = \left(\frac{\sqrt{2}-\hat{\tau}}{\sqrt{2}+\hat{\tau}}\right)^b \exp\left(\frac{1}{\sqrt{2}\gamma^2}b\hat{\tau}\right), \tag{32.55}$$

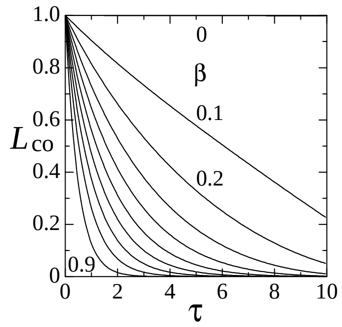


図 32.2: Comoving lumonosity for relativistic spherical flows in the RE case without heating and cooling. The values of $\beta$ are 0 to 0.9 in steps of 0.1 from top to bottom. The optical depth $\tau_c$ at the core radius is set to be 10.

where

$$b = \frac{\sqrt{2}\gamma\beta}{1+\beta^2}\tau_c. \tag{32.56}$$

The analytical solutions of the comoving luminosity (32.55) are shown in figure 2 as a function of the optical depth for several values of the flow speed. The values of $\beta$ are 0 to 0.9 in steps of 0.1.

Although the comoving luminosity (32.55) has an exponential term, the power law behavior is dominant in this case. In the non-relativistic limit of $\beta \to 0$, $b \to 0$ and the solution reduces to

$$\frac{L_0}{L_s} \sim 1 - \frac{b}{\sqrt{2}}\hat{\tau} \sim 1 - \beta\tau. \tag{32.57}$$

In the extremely relativistic limit of $\beta \to 1$, on the other hand, $b \to \gamma\tau_c/\sqrt{2}$ and the solution reduces to

$$\frac{L_0}{L_s} \sim \left(\frac{\sqrt{2}-\hat{\tau}}{\sqrt{2}+\hat{\tau}}\right)^b. \tag{32.58}$$

In contrast to this comoving luminosity, it is yet difficult to obtain analytical solutions of the spherical radiation energy density $D_0$. Even in the extremely relativistic limit, we cannot obtain the analytical solution for $D_0$.

## 32.3 Local Thermodynamic Equilibrium

Next, we consider the case of the local thermodynamic equilibrium (LTE) with a uniform source function. If the local thermodynamic equilibrium (LTE) holds in the comoving frame,

$$\frac{j_0}{4\pi} = \kappa_0 B_0, \tag{32.59}$$

where $B_0$ $(= \sigma T_0^4/\pi)$ is the frequency-integrated blackbody intensity in the comoving frame, $T_0$ being the blackbody temperature, and generally a function of the height $r$ or the optical depth $\tau$, but assumed to be constant in what follows.

In this case the relativistic moment equations (32.36) and (32.37) become

$$\begin{aligned}
&\frac{\gamma(f-\beta^2)}{f}\frac{dL_0}{d\tau} + \gamma\beta\left(\frac{1-f^2}{fr} - \frac{1}{f}\frac{df}{dr}\right)D_0\frac{dr}{d\tau} \\
&\quad +\gamma^3\left[2\beta\left(1-\frac{1}{f}\right)L_0 + (1+f)\left(1-\frac{\beta^2}{f}\right)D_0\right]\frac{d\beta}{d\tau} \\
&\quad = -\frac{\kappa_0}{\kappa_0+\sigma_0}(W_0 - D_0) - \frac{\beta}{f}L_0,
\end{aligned} \tag{32.60}$$

$$\begin{aligned}
&\gamma\left(f-\beta^2\right)\frac{dD_0}{d\tau} + \gamma\left[\frac{df}{dr} - \frac{(1-f)(1+\beta^2)}{r}\right]D_0\frac{dr}{d\tau} + 2\gamma L_0\frac{d\beta}{d\tau} \\
&\quad = L_0 + \beta\frac{\kappa_0}{\kappa_0+\sigma_0}(W_0 - D_0),
\end{aligned} \tag{32.61}$$

where

$$W_0 \equiv 16\pi^2 r^2 B_0 \tag{32.62}$$

is the spherical source function.

These equations (32.60) and (32.61) can be rearranged as

$$\begin{aligned}
&\frac{\gamma(f-\beta^2)}{f}\frac{dL_0}{d\tau} + \left[\frac{\gamma\beta}{f}\left(\frac{1-f^2}{r} - \frac{df}{dr}\right)\frac{dr}{d\tau} - \frac{\kappa_0}{\kappa_0+\sigma_0}\right]D_0 \\
&\quad +\gamma^3\left[2\beta\left(1-\frac{1}{f}\right)L_0 + (1+f)\left(1-\frac{\beta^2}{f}\right)D_0\right]\frac{d\beta}{d\tau} \\
&\quad = -\frac{\kappa_0}{\kappa_0+\sigma_0}W_0 - \frac{\beta}{f}L_0,
\end{aligned} \tag{32.63}$$

$$\gamma\left(f-\beta^2\right)\frac{dD_0}{d\tau}+\left\{\gamma\left[\frac{df}{dr}-\frac{(1-f)(1+\beta^2)}{r}\right]\frac{dr}{d\tau}+\beta\frac{\kappa_0}{\kappa_0+\sigma_0}\right\}D_0$$
$$+2\gamma L_0\frac{d\beta}{d\tau}=L_0+\beta\frac{\kappa_0}{\kappa_0+\sigma_0}W_0. \qquad (32.64)$$

These equations (32.63) and (32.64) are yet too complicated to solve analytically.

Hence, in order to simplify these equations by dropping the second terms on the left-hand sides of equations (32.63) and (32.64), we impose the two conditions:

$$\frac{\gamma\beta}{f}\left(\frac{1-f^2}{r}-\frac{df}{dr}\right)\frac{dr}{d\tau}-\frac{\kappa_0}{\kappa_0+\sigma_0} = 0, \qquad (32.65)$$
$$\gamma\left[\frac{df}{dr}-\frac{(1-f)(1+\beta^2)}{r}\right]\frac{dr}{d\tau}+\beta\frac{\kappa_0}{\kappa_0+\sigma_0} = 0. \qquad (32.66)$$

Eliminating $\kappa_0/(\kappa_0+\sigma_0)$ from equations (32.65) and (32.66), we obtain the differential equation for the variable Eddington factor $f$,

$$\frac{df}{dr}+\frac{f-1}{r}=0 \qquad (32.67)$$

as long as $dr/d\tau \neq 0$.

This equation (32.67) is easily integrated to give

$$f=1-\frac{C(\beta)}{\hat{r}}, \qquad (32.68)$$

where $C(\beta)$ is an integration constant, and generally a function of the constant flow speed $\beta$. We impose the boundary condition at the core radius such as

$$f=\frac{1+3\beta^2}{3+\beta^2} \quad \text{at} \quad r=r_{\rm c}, \qquad (32.69)$$

and the appropriate Eddington factor requested to the present case finally becomes

$$f=1-\frac{2(1-\beta^2)}{3+\beta^2}\frac{1}{\hat{r}}=1-\frac{2(1-\beta^2)}{3+\beta^2}\hat{\tau}, \qquad (32.70)$$

where $\hat{r}=r/r_{\rm c}$ and $\hat{\tau}=\tau/\tau_{\rm c}$. This variable Eddington factor (32.70) satisfies the condition: $f\to 1/3$ when $r\to r_{\rm c}$ and $\beta\to 0$, and $f\to 1$ when $r\to\infty$ ($\tau\to 0$) or $\beta\to 1$. The behavior of this Eddington factor is shown in figure 3.

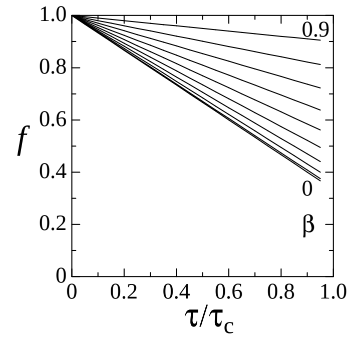


図 32.3: Eddington factor $f$ appropriate for the LTE case as a function of the optical depth $\tau$ for the various values of the flow speed $\beta$. The values of $\beta$ are 0 to 0.9 in steps of 0.1 from bottom to top.

Under these restrictive conditions, after several manipulations, equations (32.63) and (32.64) become

$$\frac{dL_0}{d\tau} = -\Gamma L_0-\Delta\frac{\kappa_0}{\kappa_0+\sigma_0}W_0, \qquad (32.71)$$
$$\frac{dD_0}{d\tau} = \frac{\Gamma}{\beta}L_0+\Gamma\frac{\kappa_0}{\kappa_0+\sigma_0}W_0, \qquad (32.72)$$

where

$$\Gamma \equiv \frac{\beta}{\gamma(f-\beta^2)}=\frac{\gamma\beta(3+\beta^2)}{3+\beta^2-2\hat{\tau}}, \qquad (32.73)$$
$$\Delta \equiv \frac{f}{\gamma(f-\beta^2)}=\gamma\frac{3+\beta^2-2(1-\beta^2)\hat{\tau}}{3+\beta^2-2\hat{\tau}}, \qquad (32.74)$$

respectively, in the present case.

Since the index $\Gamma$ is analyticall expressed by the optical depth, the solution of the homegeneous part of equation (32.71), where $W_0$ is set to be 0, is analytically obtained as

$$\frac{L_0}{L_{\rm s}}=(1-p\hat{\tau})^q, \qquad (32.75)$$

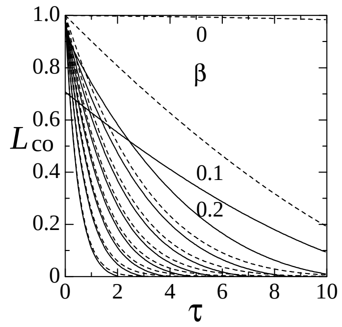


図 32.4: Comoving lumonosity for relativistic spherical flows in the LTE case with a unform source function. The values of $\beta$ are 0 to 0.9 in steps of 0.1 from top to bottom. The optical depth $\tau_c$ at the core radius is set to be 10, and the value of the source function $[\kappa_0/(\kappa_0+\sigma_0)]W_0/L_s$ is set to be unity. The solid curves represent the present analytical solutions, while the dashed ones mean the solutions of the homogeneous part.

where

$$p \equiv \frac{2}{3+\beta^2}, \tag{32.76}$$

$$q \equiv \frac{\gamma\beta(3+\beta^2)}{2}\tau_c. \tag{32.77}$$

When the spherical source function $W_0$ is uniform and $\kappa_0/(\kappa_0+\sigma_0)$ is also constant, the anlytical solution of equation (32.71) can be obtained after some manipulations as

$$\frac{L_0}{L_s} = (1-p\hat{\tau})^q + \frac{\gamma\tau_c}{1-q}\left[\left(\frac{3+\beta^2}{2}-\frac{\beta}{\gamma\tau_c}\right)-\frac{\hat{\tau}}{\gamma^2}\right]\frac{\kappa_0}{\kappa_0+\sigma_0}\frac{W_0}{L_s}. \tag{32.78}$$

The analytical solutions of the comoving luminosity (32.78) are shown in figure 4 as a function of the optical depth for several values of the flow speed. The values of $\beta$ are 0 to 0.9 in steps of 0.1.

In the LTE case the comoving luminosity (32.78) has the power law form. In the non-relativistic limit of $\beta \to 0$, $p \to 2/3$ and $q \to 3\beta\tau_c/2$, and the solution becomes a linear function of $\tau$. In the extremely relativistic limit of $\beta \to 1$, on the other hand, $p \to 1/2$ and $q \to 2\gamma\tau_c$, the solution reduces to

$$\frac{L_0}{L_s} \sim (1-p\hat{\tau})^q - \frac{\kappa_0}{\kappa_0+\sigma_0}\frac{W_0}{L_s}. \tag{32.79}$$

On the contrary to the RE case, we can obtain the analytical solutions of the spherical radiation energy density $D_0$. However, it is rather complicated and we omit the expression for $D_0$.