


相対論的輻射輸送方程式の解析解 II

共動系での相対論的*Milne-Eddington*解

平行平板＆球対称

Generalized Milne-Eddington Solution in Relativistically Moving Atmosphere

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１　動機

輻射輸送方程式の解析的な解を求める

- 非相対論的な場合でさえ非常に少ない
  - Eddington model
- 相対論的な解析解は非常に稀
  - plane-pallale　Fukue 2007, 2008a, 2008c
  - spherical　Fukue 2010a
- これらは静止系の方程式で求めた
- 今回は共動系の方程式で考えた
  - REに加えLTEの解も得た

2. 相対論的輻射輸送方程式　―般形

mixed frame

$$
\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I
= \left(\frac{\nu}{\nu_0}\right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0+\sigma_0)\, I_0 + \sigma_0 \frac{cE_0}{4\pi}\right],
$$

$$
\begin{aligned}
\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} &= \rho\gamma\,(j_0 - \kappa_0 c E_0) \\
&\quad -\rho\gamma\,(\kappa_0+\sigma_0)\,\boldsymbol{\beta}\cdot\boldsymbol{F}_0, \\
\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} &= \rho\gamma\frac{\beta^i}{c}\,(j_0 - \kappa_0 c E_0) \\
&\quad -\rho\,(\kappa_0+\sigma_0)\,\frac{\gamma-1}{\beta^2}\frac{\beta^i}{c}\,(\boldsymbol{\beta}\cdot\boldsymbol{F}_0) \\
&\quad -\frac{1}{c}\rho\,(\kappa_0+\sigma_0)\,F_0^i,
\end{aligned}
$$

２．相対論的輻射輸送方程式 鉛直方向

comoving frame

共動系での輻射強度 $I_0(z)$

鉛直座標 z

方向余弦 $\mu_0=\cos\theta_0$

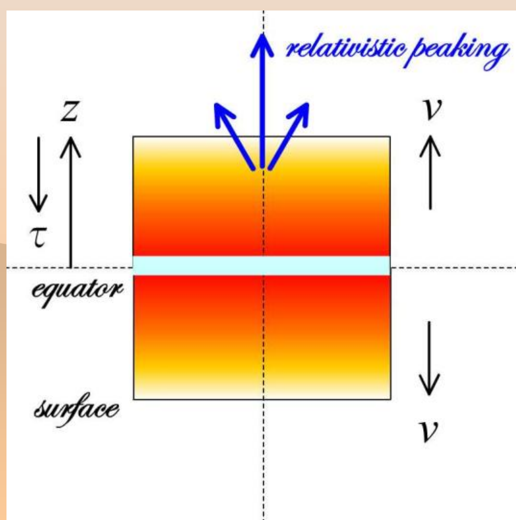
密度ρ； 不透明度κ_0，σ_0

速度 $\beta=v/c$

$$
\begin{aligned}
&\gamma(1+\beta\mu_0)\frac{1}{c}\frac{\partial I_0}{\partial t}+\gamma(\mu_0+\beta)\frac{\partial I_0}{\partial z}\\
&\quad-\gamma^3(1+\beta\mu_0)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0}-4\mu_0 I_0\right]\frac{1}{c}\frac{\partial \beta}{\partial t}\\
&\quad-\gamma^3(\mu_0+\beta)\left[(1-\mu_0^2)\frac{\partial I_0}{\partial \mu_0}-4\mu_0 I_0\right]\frac{\partial \beta}{\partial z}\\
&\quad=\rho\left[\frac{j_0}{4\pi}-(\kappa_0+\sigma_0)\,I_0+\sigma_0\frac{cE_0}{4\pi}\right].
\end{aligned}
$$

$$
\begin{aligned}
&\gamma\frac{\partial cE_0}{c\partial t}+\gamma\frac{\partial F_0}{\partial z}+\gamma\beta\frac{\partial F_0}{c\partial t}+\gamma\beta\frac{\partial cE_0}{\partial z}\\
&\quad+\gamma^3\left[2F_0+\beta(cE_0+cP_0)\right]\frac{\partial\beta}{c\partial t}\\
&\quad+\gamma^3\left[2\beta F_0+(cE_0+cP_0)\right]\frac{\partial\beta}{\partial z}\\
&\quad=\rho\,(j_0-\kappa_0 cE_0)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\gamma\frac{\partial F_0}{c\partial t}+\gamma\frac{\partial cP_0}{\partial z}+\gamma\beta\frac{\partial cP_0}{c\partial t}+\gamma\beta\frac{\partial F_0}{\partial z}\\
&\quad+\gamma^3\left[2\beta F_0+(cE_0+cP_0)\right]\frac{\partial\beta}{c\partial t}\\
&\quad+\gamma^3\left[2F_0+\beta(cE_0+cP_0)\right]\frac{\partial\beta}{\partial z}\\
&\quad=-\rho\,(\kappa_0+\sigma_0)\,F_0,
\end{aligned}
$$



2. 相対論的輻射輸送方程式 鉛直方向

comoving frame

定常

不透明度は一定

$$a \equiv \frac{\sigma_0}{\kappa_0 + \sigma_0},$$

速度は一定

光学的厚み

$$d\tau \equiv -(\kappa_0 + \sigma_0)\rho dz,$$

$$\gamma(\mu_0 + \beta)\frac{dI_0}{d\tau} = I_0 - \frac{1}{4\pi}\frac{j_0}{\kappa_0 + \sigma_0} - a\frac{cE_0}{4\pi},$$

$$\gamma\frac{dF_0}{d\tau} + \gamma\beta\frac{dcE_0}{d\tau} = -\frac{j_0}{\kappa_0 + \sigma_0} + (1-a)cE_0,$$

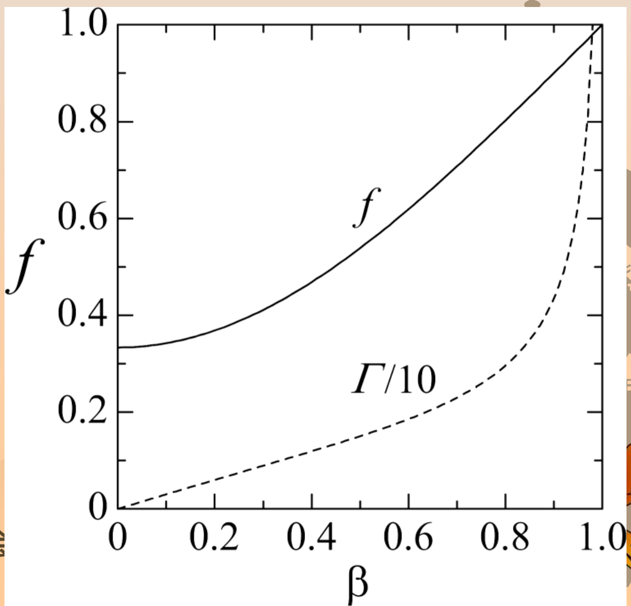
$$\gamma\frac{dcP_0}{d\tau} + \gamma\beta\frac{dF_0}{d\tau} = F_0.$$

エディントン因子

***f*(β)**

$$P_0 = f(\tau, \beta)E_0,$$

$$f(\beta) = \frac{1+3\beta^2}{3+\beta^2},$$



3. 解析解　輻射平衡の場合

- 方程式

$$\gamma \frac{dF_0}{d\tau} + \gamma\beta \frac{dcE_0}{d\tau} = 0,$$
$$\gamma \frac{dcP_0}{d\tau} + \gamma\beta \frac{dF_0}{d\tau} = F_0.$$

$$\frac{dF_0}{d\tau} = -\Gamma F_0,$$

$$\boxed{\Gamma \equiv \frac{\beta}{\gamma(f-\beta^2)}}$$

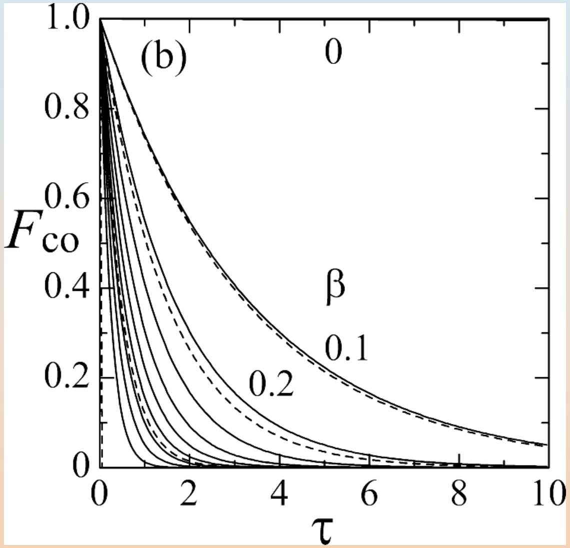
- 解析解

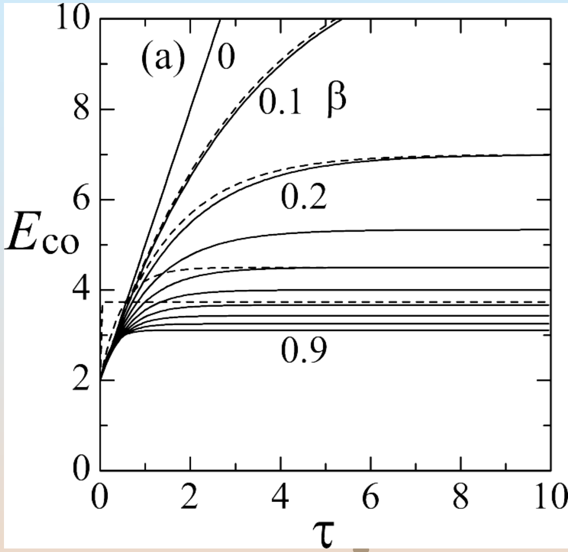
$$cP_\mathrm{s} = 2fF_\mathrm{s} \quad \text{at} \quad \tau = 0.$$

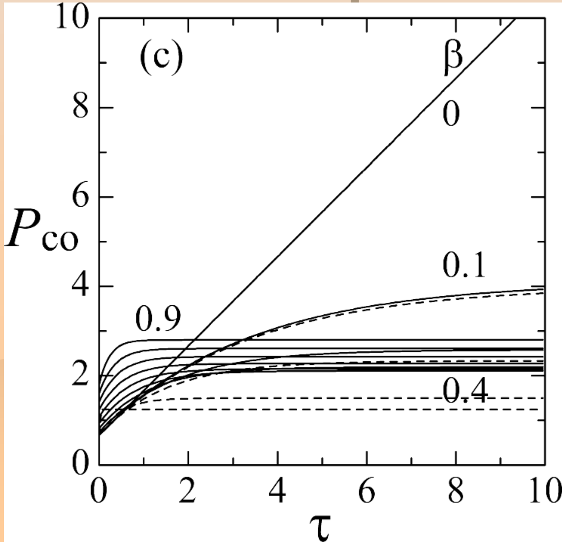
At last, the analytical solutions become

$$F_0 = F_\mathrm{s} e^{-\Gamma\tau},$$

$$cP_0 = fcE_0 = F_\mathrm{s}\frac{f}{\beta}\left(1 + 2\beta - e^{-\Gamma\tau}\right).$$

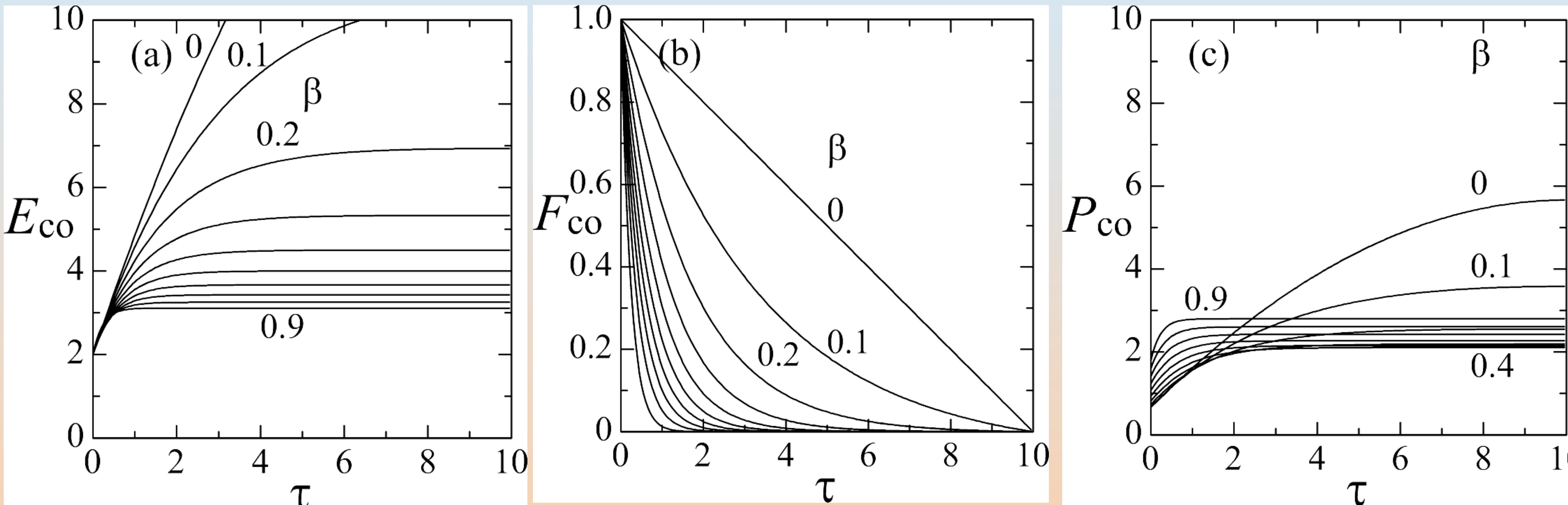


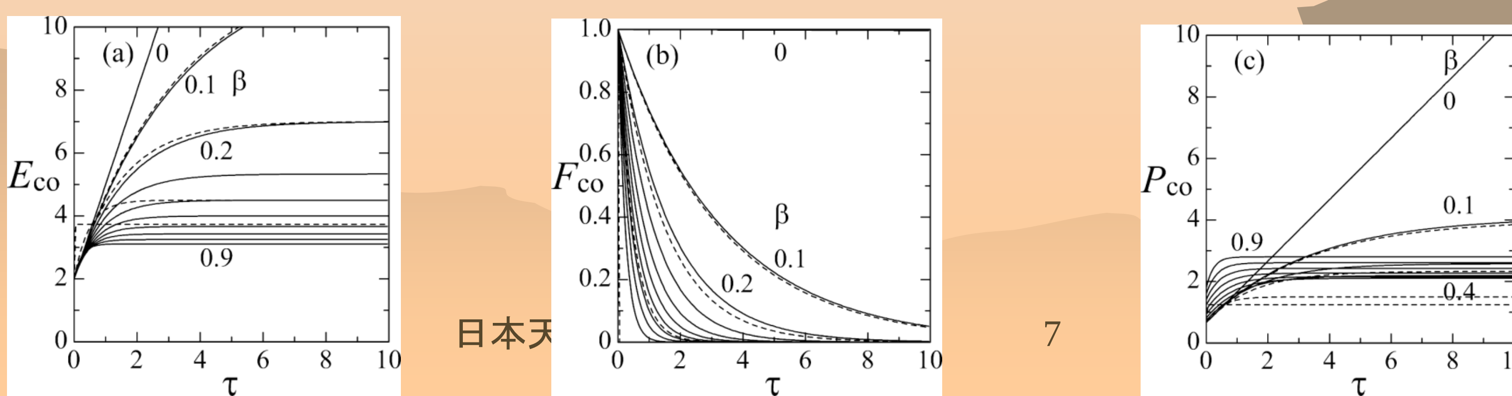




３. 解析解　輻射平衡の場合

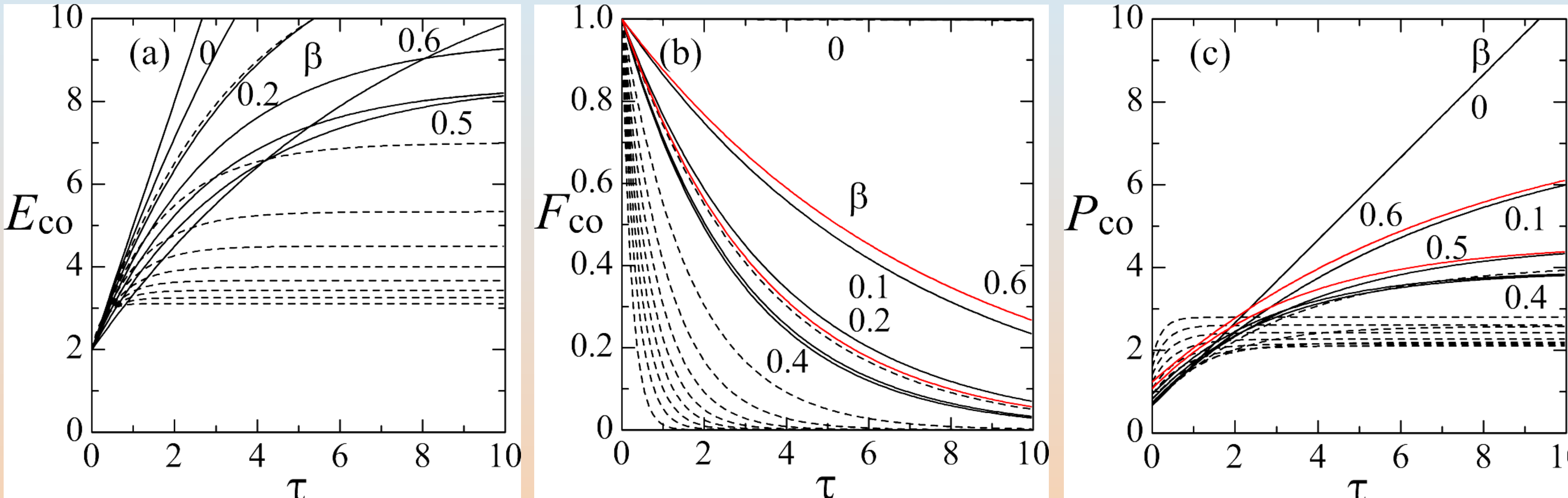
- internal heating

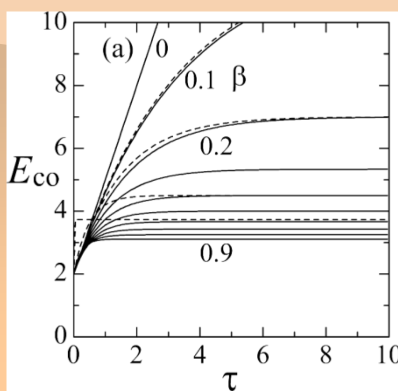


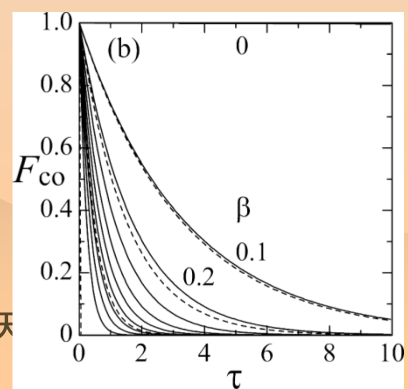


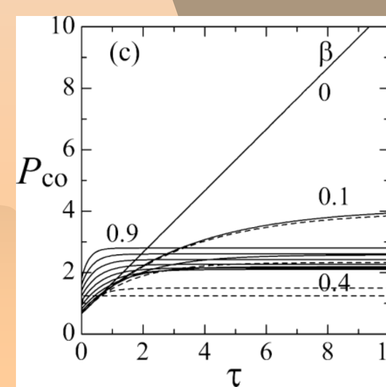
３．解析解　輻射平衡の場合

- advective cooling









3. 解析解 LTEの場合

- 方程式

$$\frac{j_0}{4\pi} = \kappa_0 B_0,$$

$$\gamma \frac{dF_0}{d\tau} + \gamma\beta \frac{dcE_0}{d\tau} = -(1-a)4\pi B_0 + (1-a)cE_0,$$

$$\gamma \frac{dcP_0}{d\tau} + \gamma\beta \frac{dF_0}{d\tau} = F_0.$$

$$\frac{d^2 F_0}{d\tau^2} + (2-a)\Gamma \frac{dF_0}{d\tau} - \frac{1-a}{\gamma\beta}\Gamma F_0$$
$$= -(1-a)\frac{f\Gamma}{\beta} 4\pi \frac{dB_0}{d\tau}.$$

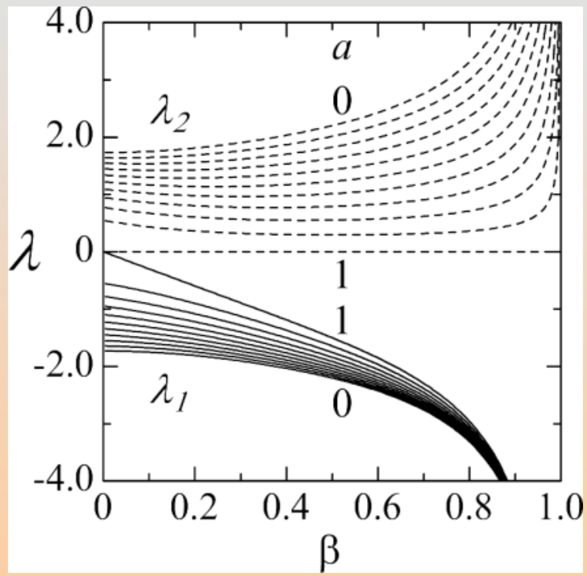
3. 解析解　LTEの場合

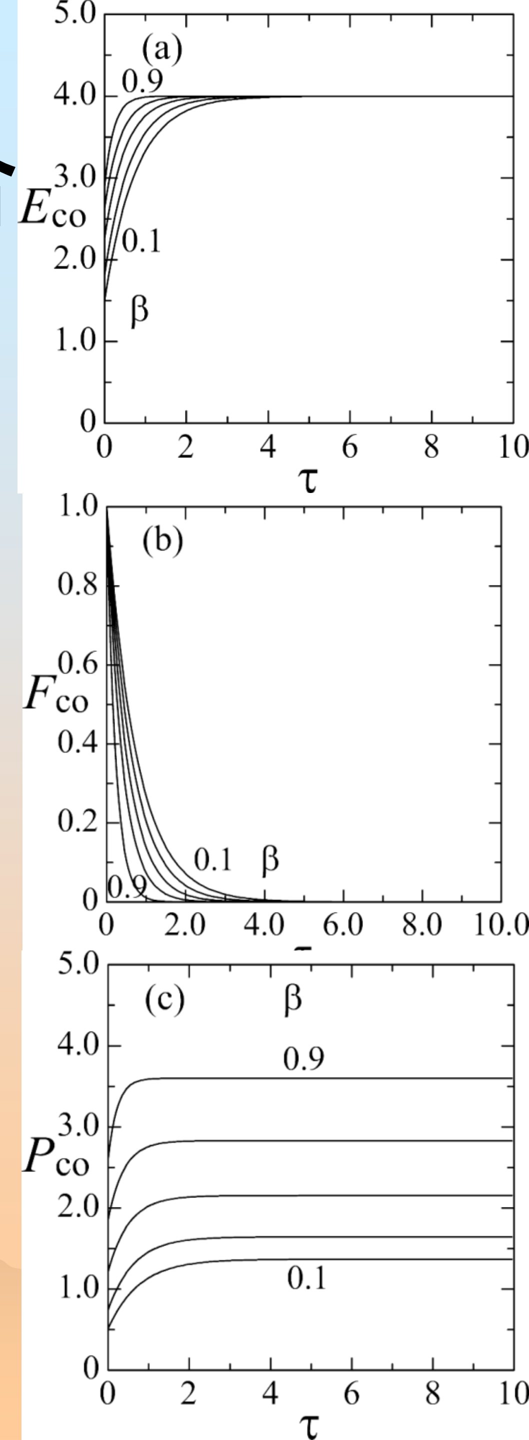
- B_0=一定

$$\frac{d^2F_0}{d\tau^2}+(2-a)\Gamma\frac{dF_0}{d\tau}-\frac{1-a}{\gamma\beta}\Gamma F_0=0.$$

$$\begin{aligned}\frac{F_0}{F_\mathrm{s}}&=\frac{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}}{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}-1}e^{\lambda_1\tau}\\&\quad-\frac{1}{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}-1}e^{\lambda_2\tau},\\\frac{cP_0}{F_\mathrm{s}}&=\frac{fcE_0}{F_\mathrm{s}}\\&=\frac{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}}{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}-1}\left(\frac{1}{\gamma\lambda_1}-\beta\right)e^{\lambda_1\tau}\\&\quad-\frac{1}{e^{(\lambda_2-\lambda_1)\tau_\mathrm{b}}-1}\left(\frac{1}{\gamma\lambda_2}-\beta\right)e^{\lambda_2\tau}\\&\quad+4f\frac{\pi B_0}{F_\mathrm{s}}.\end{aligned}$$

$$\lambda_{1,2}=-\frac{1}{2}\left[(2-a)\Gamma\pm\sqrt{(2-a)^2\Gamma^2+\frac{4(1-a)}{\gamma\beta}\Gamma}\right]$$





3. 解析解 LTEの場合

- $B_0 = B_{00} + B_1\tau$（EB近似）

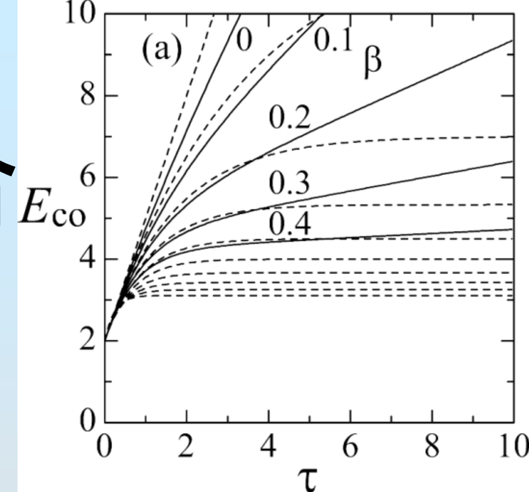
$$\frac{F_0}{F_\mathrm{s}} = 4\gamma f b_1 + (1 - 4\gamma f b_1)\, e^{\lambda_1 \tau},$$

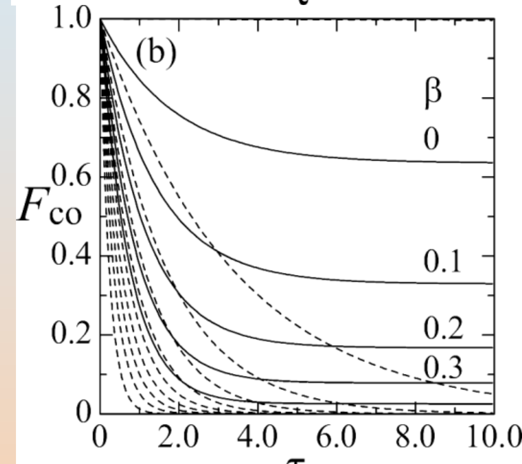
$$\frac{cP_0}{F_\mathrm{s}} = \frac{f c E_0}{F_\mathrm{s}}$$

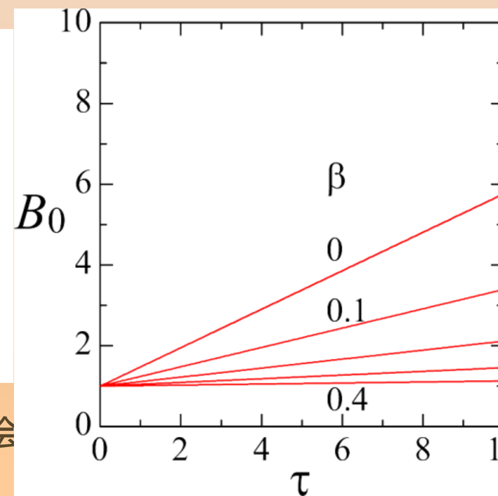
$$= 2f - \left(\frac{1}{\gamma\lambda_1} - \beta\right)(1 - 4\gamma f b_1)\left(1 - e^{\lambda_1 \tau}\right) + 4 f b_1 \tau,$$

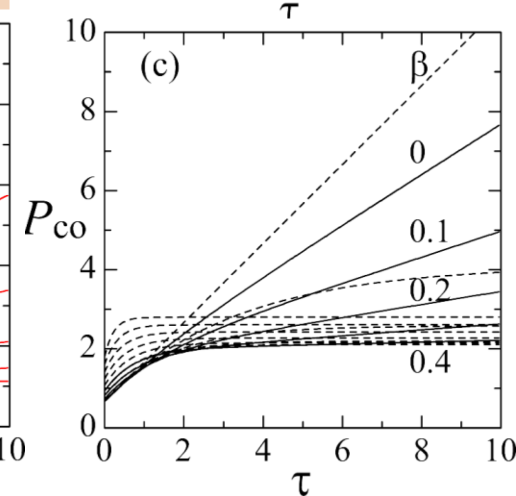
$$\lambda_1 = -\frac{1}{2}\left[(2-a)\Gamma + \sqrt{(2-a)^2\Gamma^2 + \frac{4(1-a)}{\gamma\beta}\Gamma}\right]$$

$$4\gamma f b_1 = \frac{2f - \left(\frac{1}{\gamma\lambda_1} - \beta\right) - 4 f b_0}{-\frac{1}{\gamma\lambda_1} + \frac{2-a}{1-a}\beta}.$$









2. 相対論的輻射輸送方程式　球対称

comoving frame

定常
不透明度は一定
速度は一定
光学的厚み

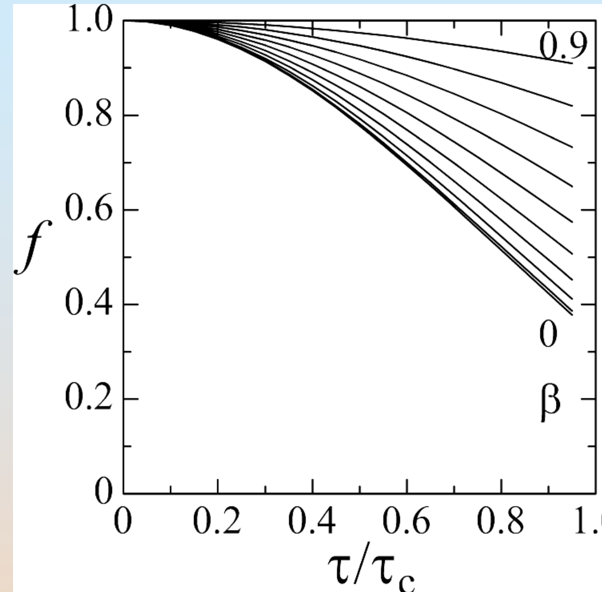
$$
\frac{\gamma(f-\beta^2)}{f}\frac{dL_0}{d\tau} + \gamma\beta\left(\frac{1-f^2}{fr} - \frac{1}{f}\frac{df}{dr}\right) D_0 \frac{dr}{d\tau}
$$
$$
+\gamma^3\left[2\beta\left(1-\frac{1}{f}\right)L_0 + (1+f)\left(1-\frac{\beta^2}{f}\right)D_0\right]\frac{d\beta}{d\tau}
$$
$$
= -\frac{4\pi r^2}{\kappa_0+\sigma_0}(j_0 - \kappa_0 c E_0) - \frac{\beta}{f}L_0,
$$

$$
\gamma\left(f-\beta^2\right)\frac{dD_0}{d\tau} + \gamma\left[\frac{df}{dr} - \frac{(1-f)(1+\beta^2)}{r}\right] D_0 \frac{dr}{d\tau}
$$
$$
+2\gamma L_0 \frac{d\beta}{d\tau}
$$
$$
= L_0 + \beta\frac{4\pi r^2}{\kappa_0+\sigma_0}(j_0 - \kappa_0 c E_0).
$$

3. 解析解　輻射平衡の場合

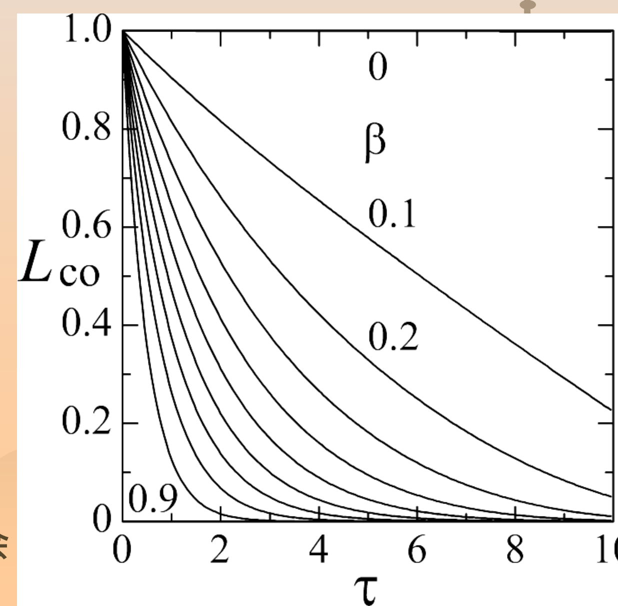
- エディントン因子$f(\tau, \beta)$

$$f = \frac{2\gamma^2(1+\beta^2)\hat{r}^2 - 1}{2\gamma^2(1+\beta^2)\hat{r}^2 + 1} = \frac{2\gamma^2(1+\beta^2) - \hat{\tau}^2}{2\gamma^2(1+\beta^2) + \hat{\tau}^2},$$



- 光度の解析解

$$\frac{L_0}{L_s} = \left(\frac{\sqrt{2} - \hat{\tau}}{\sqrt{2} + \hat{\tau}}\right)^b \exp\left(\frac{1}{\sqrt{2}\gamma^2} b\hat{\tau}\right),$$



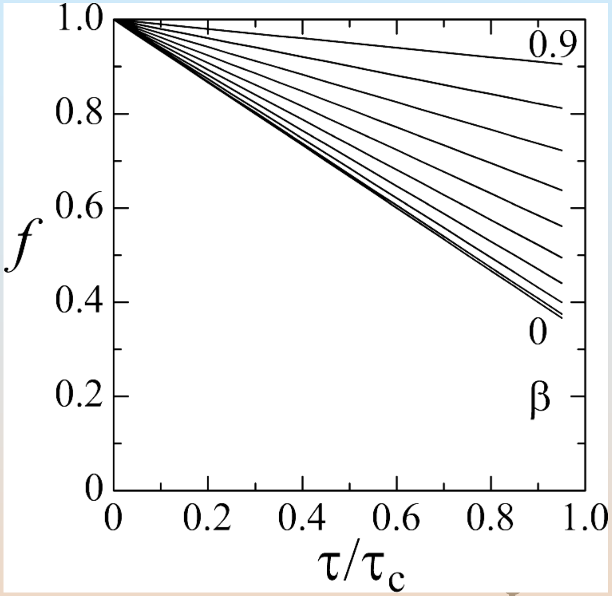
3. 解析解 LTEの場合

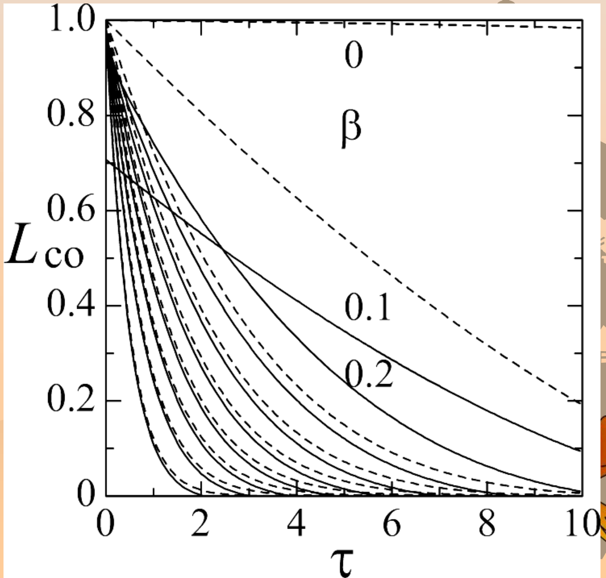
- エディントン因子$f(\tau, \beta)$

$$f = 1 - \frac{2(1-\beta^2)}{3+\beta^2}\frac{1}{\hat{r}} = 1 - \frac{2(1-\beta^2)}{3+\beta^2}\hat{\tau},$$

- 光度の解析解

$$\frac{L_0}{L_s} = (1-p\hat{\tau})^q + \frac{\gamma\tau_c}{1-q}\left[\left(\frac{3+\beta^2}{2} - \frac{\beta}{\gamma\tau_c}\right) - \frac{\hat{\tau}}{\gamma^2}\right]\frac{\kappa_0}{\kappa_0+\sigma_0}\frac{W_0}{L_s}.$$





4．まとめ　まとめ

相対論的輻射輸送方程式を解析的に解いて、
速度一定でREやLTEなどの場合に対して、
一般化されたMilne-Eddington解を求めた

解析解は、光学的厚みに関して線形的でなく、
指数的な振る舞いを示す

また、有効的な光学的厚みは、$\tau \rightarrow \Gamma\tau$となる