第1章 Radiative Transfer Equations General variables $IEFP$ in physics

After Kato, S. et al. 2008, "Black-Hole Accretion Disks"

In highly energetic astronomical systems, such as accretion disks, matter and radiation are often coupled to each other. Matter emits or absorbs radiation, while radiation gives (or removes) energy and momentum to (or from) matter. The behavior of radiation interacting with matter is known as *radiative transfer*. In this appendix we summarize the basic equations of radiative transfer in the Newtonian regime.

1.1 Radiation Fields

In astrophysics the theory of radiative transfer has been developed in the fields involving the stellar atmosphere. In this appendix we only consider a minimum of the concepts of radiative transfer. More general and detailed treatments can be found in many textbooks (e.g., Chandrasekhar 1960; Mihalas 1970; Rybicki and Lightman 1979; Mihalas and Mihalas 1984; Shu 1991; Peraiah 2002; Castor 2004).

(a) **Specific intensity and other quantities**

The *specific intensity*, $I_\nu(\boldsymbol{r}, \boldsymbol{l}, t)$ [erg s^{-1} cm^{-2} sr^{-1} Hz^{-1}], is the radiation energy carried off by the rays per unit time, unit area, unit solid angle, and unit frequency. By integrating the specific intensity over the frequency $d\nu$, we obtain the total intensity $I(\boldsymbol{r}, \boldsymbol{l}, t)$ as

$$I = \int_0^\infty I_\nu d\nu. \tag{1.1}$$

Integrating the specific intensity I_ν, multiplied by the direction cosine vector $\boldsymbol{l}$, over a solid angle $d\Omega$ (and frequency), we obtain quantities describing the radiation fields and their frequency-integrated forms, as follows:

$$\begin{aligned} E_\nu &\equiv \frac{1}{c}\int I_\nu d\Omega, & E &\equiv \int E_\nu d\nu, \\ \boldsymbol{F}_\nu &\equiv \int I_\nu \boldsymbol{l} d\Omega, & \boldsymbol{F} &\equiv \int \boldsymbol{F}_\nu d\nu, \\ P_\nu^{ij} &\equiv \frac{1}{c}\int I_\nu l^i l^j d\Omega, & P^{ij} &\equiv \int P_\nu^{ij} d\nu, \end{aligned} \tag{1.2}$$

where E_ν (E) is the radiation energy density, $\boldsymbol{F}_\nu$ ($\boldsymbol{F}$) the radiative flux, and P_ν^{ij} (P^{ij}) the radiation stress.[1]

(b) **Blackbody radiation**

Under thermodynamic equilibrium, the specific intensity becomes a Planck distribution (blackbody):

$$I_\nu(\boldsymbol{r}, \boldsymbol{l}, t) = B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp(h\nu/k_{\rm B}T) - 1}, \tag{1.3}$$

where T is the blackbody temperature, c $(= 2.9979 \times 10^{10}$ cm s$^{-1})$ the speed of light, h $(= 6.6261 \times 10^{-27}$ erg s) the Planck constant, and $k_{\rm B}$ $(= 1.3807 \times 10^{-16}$ erg K$^{-1})$ the Boltzmann constant.

In this case, for example, the frequency-integrated intensity I and the radiation energy density E become, respectively,

$$I(\boldsymbol{r}, \boldsymbol{l}, t) = B(T) = \frac{1}{\pi}\sigma T^4, \tag{1.4}$$

$$E(\boldsymbol{r}, t) = \frac{4\pi}{c} B(T) = aT^4, \tag{1.5}$$

[1] Often used are the mean intensity J_ν, the Eddington flux H_ν, and the mean pressure stress K_ν. They are related to the one-dimensional components of E_ν, F_ν^i, and P_ν^{ij} by

$$J_\nu = \frac{c}{4\pi}E_\nu, \quad H_\nu = \frac{1}{4\pi}F_\nu, \quad K_\nu = \frac{c}{4\pi}P_\nu.$$

where σ (= 5.6705×10^{-5} erg s^{-1} cm^{-2} K^{-4}) is the Stefan-Boltzmann constant and a (= 7.5660×10^{-15} erg cm^{-3} K^{-4}) is the radiation constant; $\sigma = ac/4$. Furthermore, at the surface of a blackbody radiator, such as a star or an accretion disk, the outward radiative flux F becomes

$$F(\boldsymbol{r},t) = \pi B(T) = \sigma T^4. \tag{1.6}$$

1.2 Equations of Radiative Transfer

We first derive the basic equations describing the behavior of the radiation fields interacting with matter.

1.2.1 Transfer Equation

A change in the specific intensity is expressed by the *transfer equation*, which is equivalent to the Boltzmann equation for matter.

By means of the mass emissivity (i.e., emissivity per unit mass) j_ν [erg s^{-1} g^{-1} sr^{-1} Hz^{-1}], the mass absorption coefficient (i.e., absorption cross-section per unit mass) κ_ν [cm^2 g^{-1}] (= $\kappa_\nu^{\rm abs}+\kappa_\nu^{\rm sca}$, where $\kappa_\nu^{\rm abs}$ is for the true absorption and $\kappa_\nu^{\rm sca}$ for scattering), the *transfer equation* is expressed as

$$\begin{aligned}\frac{1}{c}\frac{\partial I_\nu(\boldsymbol{l})}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu(\boldsymbol{l}) &= \frac{1}{4\pi}\rho j_\nu - \rho\kappa_\nu^{\rm abs} I_\nu(\boldsymbol{l}) \\ &-\rho\kappa_\nu^{\rm sca}\int \phi_\nu(\boldsymbol{l},\boldsymbol{l}')I_\nu(\boldsymbol{l})d\Omega' + \rho\kappa_\nu^{\rm sca}\int \phi_\nu(\boldsymbol{l},\boldsymbol{l}')I_\nu(\boldsymbol{l}')d\Omega', \end{aligned}\tag{1.7}$$

where $\phi_\nu(\boldsymbol{l},\boldsymbol{l}')$ is the scattering probability function ($\int \phi_\nu d\Omega = 1$).

If the scattering is isotropic, $\phi_\nu = 1/4\pi$ and the transfer equation (2.3) becomes

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu = \frac{1}{4\pi}\rho j_\nu - \rho\left(\kappa_\nu^{\rm abs}+\kappa_\nu^{\rm sca}\right) I_\nu + \rho\kappa_\nu^{\rm sca}\frac{c}{4\pi}E_\nu. \tag{1.8}$$

This equation is an integro-differential equation on I_ν, and generally too difficult to obtain precise solutions.

1.2.2 Moment Equations

We often take *moments* of the transfer equation, since it has too much information to directly solve.[2] Integrating the transfer equation (1.8) over a solid angle, we obtain the zeroth moment. Integrating it over a solid angle, after being multiplied by the direction cosine, we obtain the first moment. The zeroth and first moments of equation (1.8) are, respectively,

$$\frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^k}{\partial x^k} = \rho\left(j_\nu - c\kappa_\nu^{\rm abs}E_\nu\right), \tag{1.9}$$

$$\frac{1}{c^2}\frac{\partial F_\nu^i}{\partial t} + \frac{\partial P_\nu^{ik}}{\partial x^k} = -\frac{\rho(\kappa_\nu^{\rm abs}+\kappa_\nu^{\rm sca})}{c}F_\nu^i. \tag{1.10}$$

The former corresponds to the energy conservation of radiation with the energy exchange with matter, whereas the latter corresponds to the momentum conservation of radiation with the momentum loss to matter.

Moreover, integrating equations (1.8), (1.9), and (1.10) over the frequency, we obtain a frequency-integrated transfer equation and its moment equations:

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I = \rho\left(\frac{j}{4\pi} - \bar{\kappa}_I I + \bar{\kappa}_E^{\rm sca}\frac{c}{4\pi}E\right), \tag{1.11}$$

$$\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} = \rho\left(j - c\bar{\kappa}_E^{\rm abs}E\right), \tag{1.12}$$

$$\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} = -\frac{1}{c}\rho\bar{\kappa}_F F^i, \tag{1.13}$$

where

$$j \equiv \int j_\nu d\nu, \tag{1.14}$$

$$\bar{\kappa}_I \equiv \frac{1}{I}\int\left(\kappa_\nu^{\rm abs}+\kappa_\nu^{\rm sca}\right)I_\nu d\nu, \tag{1.15}$$

$$\bar{\kappa}_E^{\rm abs} \equiv \frac{1}{E}\int \kappa_\nu^{\rm abs}E_\nu d\nu, \tag{1.16}$$

$$\kappa_E^{\rm sca} \equiv \frac{1}{E}\int \kappa_\nu^{\rm sca}E_\nu d\nu, \tag{1.17}$$

$$\bar{\kappa}_F \equiv \frac{1}{F^i}\int\left(\kappa_\nu^{\rm abs}+\kappa_\nu^{\rm sca}\right)F_\nu^i d\nu. \tag{1.18}$$

[2] For matter we usually use the hydrodynamical equations instead of the Boltzmann equation.

It should be noted that

$$f^i_{\rm rad} = \frac{\rho}{c}\int\left(\kappa^{\rm abs}_\nu + \kappa^{\rm sca}_\nu\right)F^i_\nu d\nu = \frac{1}{c}\rho\bar{\kappa}_F F^i \tag{1.19}$$

is the radiative force per unit volume, acting on matter.

1.2.3 Closure Relation (Eddington Approximation)

The zeroth-moment equation (1.9) contains the radiative flux, which is determined by the first-moment equation (1.10). Furthermore, the first-moment equation (1.10) contains the radiation stress, which is determined by the second-moment equation. This means that in order to solve the moment equations we need some relation to close the sequence. As a closure relation, we often adopt the *Eddington approximation*:

$$P^{ij}_\nu = \frac{\delta^{ij}}{3}E_\nu. \tag{1.20}$$

This approximation is valid when the radiation fields are almost *isotropic*.

In the case of a flat-disk configuration, this relation holds with good accuracy in an optically thin regime as well as in an optically thick one.[3] In general cases of a two-dimensional configuration, such as geometrically thick disks, however, this relation would not be adequate in an optically thin regime, and alternative closure relation is necessary (see section D.3).

1.2.4 Rosseland Approximation

When the medium is sufficiently *optically thick*, as well as the radiation isotropy, we may use the *diffusion approximation* (Rosseland approximation).

In an optically thick regime local thermodynamic equilibrium (LTE),

$$j_\nu = 4\pi\kappa^{\rm abs}_\nu B_\nu(T), \tag{1.21}$$

holds, while equation (1.9) can be approximated as $j_\nu = c\kappa^{\rm abs}_\nu E_\nu$ as long as the radiation intensity is a slowly varying function of space and time. Hence,

$$E_\nu = \frac{4\pi}{c}B_\nu \quad \text{and} \quad P^{ij}_\nu = \frac{\delta^{ij}}{3}\frac{4\pi}{c}B_\nu. \tag{1.22}$$

In the steady state, from equation (1.10), we thus obtain

$$\begin{aligned} F^i_\nu &= -\frac{c}{\rho(\kappa^{\rm abs}_\nu + \kappa^{\rm sca}_\nu)}\frac{\partial P^{ik}}{\partial x^k} \\ &= -\frac{4\pi}{3\rho(\kappa^{\rm abs}_\nu + \kappa^{\rm sca}_\nu)}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial x^i}. \end{aligned} \tag{1.23}$$

This means that the radiation energy is transported by an isotropic diffusion of photons.

Integrating equation (1.23) over the frequency, we obtain

$$\boldsymbol{F} = \int \boldsymbol{F}_\nu d\nu = -\frac{4\pi}{3\rho}\boldsymbol{\nabla}T\frac{\displaystyle\int\frac{1}{\kappa^{\rm abs}_\nu + \kappa^{\rm sca}_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\displaystyle\int\frac{\partial B_\nu}{\partial T}d\nu}\int\frac{\partial B_\nu}{\partial T}d\nu. \tag{1.24}$$

Since

$$\int\frac{\partial B_\nu}{\partial T}d\nu = \frac{d}{dT}\int B_\nu d\nu = \frac{d}{dT}\left(\frac{1}{\pi}\sigma T^4\right) = \frac{4}{\pi}\sigma T^3, \tag{1.25}$$

we finally obtain a frequency-integrated radiative flux,

$$\boldsymbol{F} = -\frac{4acT^3}{3\bar{\kappa}_{\rm R}\rho}\boldsymbol{\nabla}T, \tag{1.26}$$

where $\bar{\kappa}_{\rm R}$ is the Rosseland mean opacity:

$$\frac{1}{\bar{\kappa}_{\rm R}} \equiv \frac{\displaystyle\int_0^\infty\frac{1}{\kappa^{\rm abs}_\nu + \kappa^{\rm sca}_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\displaystyle\int_0^\infty\frac{\partial B_\nu}{\partial T}d\nu}. \tag{1.27}$$

Equation (1.26) is often called a radiative conduction equation, where the effective 'conductivity' is $4acT^3/3\bar{\kappa}_{\rm R}\rho$, which is inversely proportional to the opacity.

[3] In the spherical case $P^{rr}_\nu = E_\nu$, while all other components vanish in an optically thin regime.

The Rosseland mean opacities for free-free and bound-free absorptions, $\kappa_{\rm ff}$ and $\kappa_{\rm bf}$, are approximately expressed by Kramers' law:

$$\begin{aligned}
\kappa_{\rm ff} &= 3.68 \times 10^{22} g_{\rm ff}(X+Y)(1+X)\rho T^{-7/2} \ \ {\rm cm^2\ g^{-1}} \\
&\sim 6.24 \times 10^{22} \rho T^{-7/2} \ \ {\rm cm^2\ g^{-1}}, \qquad (1.28) \\
\kappa_{\rm bf} &= 4.34 \times 10^{25} (g_{\rm bf}/t) Z (1+X)\rho T^{-7/2} \ \ {\rm cm^2\ g^{-1}} \\
&\sim 1.50 \times 10^{24} \rho T^{-7/2} \ \ {\rm cm^2\ g^{-1}}, \qquad (1.29)
\end{aligned}$$

where $g_{\rm ff}$ and $g_{\rm bf}$ are the mean Gaunt factors, of order unity, for the free-free and bound-free transitions, respectively, t is the guillotine factor of order unity, and X, Y, and Z are the abundances of hydrogen, helium, and metal, respectively (Morse 1940; Schwarzschild 1958). In the low metalicity case the free-free absorption dominates the bound-free one, while the bound-free absorption will dominate the free-free one in the high metalicity case (Schwarzschild 1958).

In addition, the electron scattering opacity is given by

$$\kappa_{\rm es} = 0.20(1+X) \ \ {\rm cm^2\ g^{-1}} \sim 0.4 \ \ {\rm cm^2\ g^{-1}}. \qquad (1.30)$$

1.2.5 Source Function

When the transfer equation (1.8) is expressed as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l} \cdot \boldsymbol{\nabla}) I_\nu = \rho \left(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}\right)(S_\nu - I_\nu), \qquad (1.31)$$

the *source function* S_ν is introduced as

$$S_\nu \equiv \frac{1}{\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}} \left(\frac{j_\nu}{4\pi} + \frac{c\kappa_\nu^{\rm sca}}{4\pi} E_\nu \right). \qquad (1.32)$$

In the case of the local thermodynamic equilibrium (LTE), the source function (1.32) becomes

$$\begin{aligned}
S_\nu &= \frac{1}{\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}} \left(\kappa_\nu^{\rm abs} B_\nu + \frac{c\kappa_\nu^{\rm sca}}{4\pi} E_\nu \right) \\
&= (1 - A_\nu) B_\nu + A_\nu \frac{c}{4\pi} E_\nu, \qquad (1.33)
\end{aligned}$$

where

$$A_\nu = \frac{\kappa_\nu^{\rm sca}}{\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}} \qquad (1.34)$$

is the *single scattering albedo*.

In terms of this source function, for example, equation (1.8) and (1.9), and (1.10) with the Eddington approximation (1.20) are, respectively, reexpressed as

$$\begin{aligned}
\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l} \cdot \boldsymbol{\nabla}) I_\nu &= \rho \left(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}\right)(S_\nu - I_\nu), \qquad (1.35) \\
\frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^k}{\partial x^k} &= \rho \left(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}\right)(4\pi S_\nu - cE_\nu), \qquad (1.36) \\
\frac{1}{c^2}\frac{\partial F_\nu^i}{\partial t} + \frac{1}{3}\frac{\partial E_\nu}{\partial x^i} &= -\frac{\rho(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca})}{c} F_\nu^i. \qquad (1.37)
\end{aligned}$$

1.3 Optically Thick to Thin Regimes

The Eddington approximation (1.20) as a closure relation holds when the radiation field is *nearly isotropic*.[4] In the optically thin regime, or in the case where the anisotropy of radiation becomes important, we should carefully treat the Eddington approximation.[5] Indeed, in the particular configuration of plane-parallel flat disks, the Eddington approximation (1.20) holds, but it would be violated in general configurations. In this part, we briefly show the treatment in such general cases.

1.3.1 Variable Eddington Factor

Generalization of the Eddington approximation (1.20) is useful in semi-analytical cases,

$$P^{ij} = f^{ij} E, \qquad (1.38)$$

[4] The Rosseland approximation holds when the medium is sufficiently optically thick and the photon mean-free path is sufficiently smaller than the typical scale, and when the velocity gradient is sufficiently small and the local diffusion is isotropic.

[5] In the relativistic regime, where the flow speed is on the order of the speed of light, we also carefully treat the closure relation. See appendix E.

where f^{ij} is the Eddington tensor, and is generally a function of the optical depth τ. This relation (1.38) is reduced to equation (1.20), if we assume the radiation field is isotropic: $f^{ij} = \delta^{ij}/3$.

Suitable forms of the Eddington tensor are adopted in each problem and configuration. For example, in the spherically symmetric case, the diagonal part of the Eddington tensor is often set as

$$\left(f(\tau), \frac{1}{2} - \frac{1}{2}f(\tau), \frac{1}{2} - \frac{1}{2}f(\tau) \right), \tag{1.39}$$

where

$$f(\tau) = \frac{1+\tau}{1+3\tau} \tag{1.40}$$

is a *variable Eddington factor* (Tamazawa et al. 1975; see figure D.1). In the plane-parallel case, on the other hand, we can use $f = 1/3$, even in the optically thin regime for a static atmosphere.

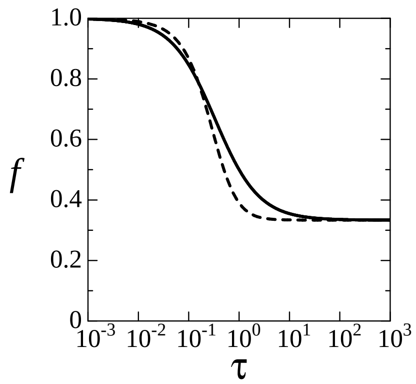

図 1.1: Variable Eddington factors as a function of the optical depth τ. A thick solid curve denotes a variable Eddington factor (1.40), while a thick dashed one means a variable Eddington factor (1.46), where R_ν is read as τ^{-1}.

1.3.2 Flux-Limited Diffusion

The diffusion (Rosseland) approximation implies that in a steady state the radiation energy is transported by an isotropic diffusion of photons:

$$\boldsymbol{F} = -\frac{c}{3\bar{\kappa}_{\rm R}\rho}\boldsymbol{\nabla} E \tag{1.41}$$

[see equation (1.26)]. This gives the correct flux in an optically thick regime, where the photon mean-free path of $\sim 1/(\bar{\kappa}_{\rm R}\rho)$ is sufficiently smaller than the typical scale for the change of E. In an optically thin regime, where the mean-free path diverges, on the other hand, this flux tends to infinity. Such a situation, however, is unphysical, since the rate at which radiation transports energy is finite, even in an optically thin regime. That is, the magnitude of the flux can be no greater than the radiation energy density times the maximum transport speed.[6] Namely, the radiative flux $\boldsymbol{F}$ should be *limited* in the optically thin regime in some way, which is the *flux-limited diffusion theory* (Levermore and Pomraning 1981; Pomraning 1983 for a relativistic correction; Melia and Zylstra 1991 for a scattering medium; Anile and Romano 1992 for a covariant form; Turner and Stone 2001 for a numerical calculation).

(a) **General forms**

In the flux-limited diffusion (FLD) theory, we also assume Fick's law of diffusion for radiation:

$$\boldsymbol{F}_\nu = -\frac{c\lambda_\nu}{\kappa_\nu\rho}\boldsymbol{\nabla} E_\nu, \tag{1.42}$$

where $\kappa_\nu = \kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}$. Here, we introduce the *flux limiter* $\lambda_\nu(E_\nu)$, which is restricted in the range of 0 (thin) $\leq \lambda_\nu \leq 1/3$ (thick). The appropriate form of this flux limiter is given later.

Using the flux limiter λ_ν, we obtain (Levermore and Pomraning 1981)

$$P_\nu^{ij} = f_\nu^{ij} E_\nu, \tag{1.43}$$

[6] In a spherical case the flux is limited as $|\boldsymbol{F}| \leq cE$, while it is limited as $|\boldsymbol{F}| \leq cE/2$ in a plane-parallel case.

where the Eddington tensor f_ν^{ij} is expressed as

$$f_\nu^{ij} = \frac{1}{2}\left(1 - f_\nu\right)\delta^{ij} + \frac{1}{2}\left(3f_\nu - 1\right)n^i n^j. \tag{1.44}$$

In this equation (1.44),

$$n^i \equiv \frac{\boldsymbol{\nabla} E_\nu}{|\boldsymbol{\nabla} E_\nu|} \tag{1.45}$$

is the unit vector in the direction of the radiation energy density gradient, i.e., the radiative flux, which is determined by the local radiation field. Furthermore, the Eddington factor $f_\nu(E_\nu)$ is expressed as

$$f_\nu = \lambda_\nu + \lambda_\nu^2 R_\nu^2, \tag{1.46}$$

where

$$R_\nu \equiv \frac{|\boldsymbol{\nabla} E_\nu|}{\kappa_\nu \rho E_\nu} \tag{1.47}$$

is the optical depth parameter, since $R_\nu \sim 1/\tau$, and is also determined by the local quantities.

Thus, if we give some appropriate form of λ_ν, all relations are fixed by the local quantities. As a choice of λ_ν, Levermore and Pomraning (1981) proposed a relation,

$$\lambda_\nu = \frac{2 + R_\nu}{6 + 3R_\nu + R_\nu^2}, \tag{1.48}$$

although many other choices are possible, which preserve causality and are consistent with the assumption of smoothness in the radiation field.[7]

In the optically thick limit ($R_\nu \to 0$), we find $\lambda_\nu \to 1/3$ and $f_\nu \to 1/3$. In the optically thin limit ($R_\nu \to \infty$), on the other hand, we have $\lambda_\nu \to 1/R_\nu$ and $f_\nu \to 1 - 1/R_\nu$.

(b) **Vertical case**

[7]For example, we quote two of them (Castor 2004):

$$\lambda_\nu = \frac{3}{3 + R_\nu}, \qquad \lambda_\nu = \frac{1}{R_\nu}\left(\coth R_\nu - \frac{1}{R_\nu}\right).$$

For the problem of an accretion disk concentrating to the vertical direction, the flux-limited diffusion approximation is expressed as

$$F_\nu^z = -\frac{c\lambda_\nu}{\kappa_\nu \rho}\frac{\partial E_\nu}{\partial z}, \tag{1.49}$$

$$P_\nu^{rr} = P_\nu^{\varphi\varphi} = \frac{1}{2}(1 - f_\nu)E_\nu \quad \text{and} \quad P_\nu^{zz} = f_\nu E_\nu. \tag{1.50}$$

In the optically thick diffusion limit we have $F_\nu^z = -(c/3\kappa_\nu\rho)\partial E_\nu/\partial z$ and $P_\nu^{rr} = P_\nu^{\varphi\varphi} = P_\nu^{zz} = E_\nu/3$, while in the optically thin streaming limit we have $|F_\nu^z| = cE_\nu$, $P_\nu^{rr} = P_\nu^{\varphi\varphi} = 0$, and $P_\nu^{zz} = E_\nu$. These give correct relations in the optically thick and thin limits, respectively.

1.4 Matter Coupling

The radiative force exerting on matter per unit mass is, from equation (1.13),

$$-\frac{1}{\rho}\left(\frac{1}{c^2}\frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k}\right) = \frac{\bar{\kappa}_F}{c}F^i, \tag{1.51}$$

while the net energy transfer rate to matter per unit mass is, from equation (1.12),

$$-\frac{1}{\rho}\left(\frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k}\right) = -j + c\bar{\kappa}_E^{\rm abs}E. \tag{1.52}$$

Thus, under the present approximation, the equation of motion and the energy equation for matter are written as, respectively,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\boldsymbol{\nabla}\psi - \frac{1}{\rho}\boldsymbol{\nabla}p + \frac{\bar{\kappa}_F}{c}\boldsymbol{F}, \tag{1.53}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)e + \frac{p}{\rho}\boldsymbol{\nabla v} = \frac{1}{\rho}q^+ - j + c\bar{\kappa}_E^{\rm abs}E, \tag{1.54}$$

where $\boldsymbol{v}$ is the velocity, ψ the gravitational potential, p the pressure, e the internal energy per unit mass, and q^+ the (viscous) heating rate per unit volume. For an ideal gas, the internal energy is expressed as $e = [1/(\gamma-1)](p/\rho)$, and the energy equation (1.54) is rewritten as

$$\frac{1}{\gamma - 1}\left(\frac{dp}{dt} - \gamma\frac{p}{\rho}\frac{d\rho}{dt}\right) = q^+ - \rho(j - c\bar{\kappa}_E^{\rm abs}E). \tag{1.55}$$

1.5 Plane-Parallel Expression

For a static atmosphere in the plane-parallel geometry (z), the hydrodynamic equations and transfer equations become as follows.

For matter, the vertical momentum balance and energy equation are, respectively,

$$0 = -\rho\frac{d\psi}{dz} - \frac{dp}{dz} + \frac{\bar{\kappa}_F}{c}\rho F, \tag{1.56}$$

$$0 = q_{\rm vis}^{+} - \rho\left(j - c\bar{\kappa}_E^{\rm abs}E\right), \tag{1.57}$$

where ψ is the gravitational potential, p the gas pressure, and $q_{\rm vis}^{+}$ the viscous-heating rate per unit volume. The opacities are assumed to be independent of the frequency (gray approximation). Under the α prescription, the viscous-heating rate is proportional to the pressure, and therefore may depend on z.

For radiation, the frequency-integrated transfer equation (1.11), the zeroth moment equation (1.9), and the first moment equation (1.10) become, respectively,

$$\cos\theta\frac{dI}{dz} = \rho\left(\frac{j}{4\pi} - \bar{\kappa}_I I + \bar{\kappa}_E^{\rm sca}\frac{c}{4\pi}E\right), \tag{1.58}$$

$$\frac{dF}{dz} = \rho\left(j - c\bar{\kappa}_E^{\rm abs}E\right), \tag{1.59}$$

$$\frac{dP}{dz} = -\frac{1}{c}\rho\bar{\kappa}_F, \tag{1.60}$$

where I is the frequency-integrated specific intensity, E the radiation energy density, F the vertical component of the radiative flux, P the zz-component of the radiation stress tensor, and θ the polar angle. The mass emissivity j and opacities are assumed to be independent of the frequency (gray approximation).

Application to standard disks are discussed in subsection 3.2.9.

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2.1 Radiation Fields

(a) **Specific intensity and other quantities**

The *specific intensity*, $I_\nu(\boldsymbol{r}, \boldsymbol{l}, t)$ [erg s^{-1} cm^{-2} sr^{-1} Hz^{-1}], is the radiation energy carried off by the rays per unit time, unit area, unit solid angle, and unit frequency. By integrating the specific intensity over the frequency $d\nu$, we obtain the total intensity $I(\boldsymbol{r}, \boldsymbol{l}, t)$ as

$$I = \int_0^\infty I_\nu d\nu. \tag{2.1}$$

Integrating the specific intensity I_ν, multiplied by the direction cosine vector $\boldsymbol{l}$, over a solid angle $d\Omega$ (and frequency), we obtain quantities describing the radiation fields and their frequency-integrated forms, as follows:

$$\begin{aligned} J_\nu &\equiv \frac{1}{4\pi}\int I_\nu d\Omega, & J &\equiv \int J_\nu d\nu, \\ \boldsymbol{H}_\nu &\equiv \frac{1}{4\pi}\int I_\nu \boldsymbol{l} d\Omega, & \boldsymbol{H} &\equiv \int \boldsymbol{H}_\nu d\nu, \\ K_\nu^{ij} &\equiv \frac{1}{4\pi}\int I_\nu l^i l^j d\Omega, & K^{ij} &\equiv \int K_\nu^{ij} d\nu, \end{aligned} \tag{2.2}$$

where J_ν is the *mean intensity*, $\boldsymbol{H}_\nu$ the *Eddington flux*, and K_ν^{ij} the *K-integral* (in plane-parallel).

2.2 Equations of Radiative Transfer

We first derive the basic equations describing the behavior of the radiation fields interacting with matter.

2.2.1 Transfer Equation

A change in the specific intensity is expressed by the *transfer equation*, which is equivalent to the Boltzmann equation for matter.

By means of the mass emissivity (i.e., emissivity per unit mass) j_ν [erg s^{-1} g^{-1} sr^{-1} Hz^{-1}] the mass absorption coefficient (i.e., absorption cross-section per unit mass) κ_ν [cm^2 g^{-1}] ($= \kappa_\nu + \sigma_\nu$, where κ_ν is for the true absorption and σ_ν for scattering), the *transfer equation* is expressed as

$$\begin{aligned} \frac{1}{c}\frac{\partial I_\nu(\boldsymbol{l})}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu(\boldsymbol{l}) &= \frac{1}{4\pi}\rho j_\nu - \rho\kappa_\nu I_\nu(\boldsymbol{l}) \\ &-\rho\sigma_\nu \int \phi_\nu(\boldsymbol{l}, \boldsymbol{l}')I_\nu(\boldsymbol{l})d\Omega' + \rho\sigma_\nu \int \phi_\nu(\boldsymbol{l}, \boldsymbol{l}')I_\nu(\boldsymbol{l}')d\Omega', \end{aligned} \tag{2.3}$$

where $\phi_\nu(\boldsymbol{l}, \boldsymbol{l}')$ is the scattering probability function ($\int \phi_\nu d\Omega = 1$).

If the scattering is isotropic, $\phi_\nu = 1/4\pi$ and the transfer equation (**??**) becomes

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu = \frac{1}{4\pi}\rho j_\nu - \rho\left(\kappa_\nu + \sigma_\nu\right) I_\nu + \rho\sigma_\nu J_\nu. \tag{2.4}$$

This equation is an integro-differential equation on I_ν, and generally too difficult to obtain precise solutions.

2.2.2 Moment Equations

We often take *moments* of the transfer equation, since it has too much information to directly solve.[1] Integrating the transfer equation (2.4) over a solid

[1] For matter we usually use the hydrodynamical equations instead of the Boltzmann equation.

angle, we obtain the zeroth moment. Integrating it over a solid angle, after being multiplied by the direction cosine, we obtain the first moment. The zeroth and first moments of equation (2.4) are, respectively,

$$\frac{\partial J_\nu}{c\partial t} + \frac{\partial H_\nu^k}{\partial x^k} = \rho\left(\frac{j_\nu}{4\pi} - \kappa_\nu J_\nu\right), \tag{2.5}$$

$$\frac{\partial H_\nu^i}{c\partial t} + \frac{\partial K_\nu^{ik}}{\partial x^k} = -\rho(\kappa_\nu + \sigma_\nu)H_\nu^i. \tag{2.6}$$

The former corresponds to the energy conservation of radiation with the energy exchange with matter, whereas the latter corresponds to the momentum conservation of radiation with the momentum loss to matter.

Moreover, integrating equations (2.4), (2.5), and (2.6) over the frequency, we obtain a frequency-integrated transfer equation and its moment equations:

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I = \rho\left(\frac{j}{4\pi} - \bar{\kappa}_I I + \bar{\sigma}_J J\right), \tag{2.7}$$

$$\frac{\partial J}{c\partial t} + \frac{\partial H^k}{\partial x^k} = \rho\left(\frac{j}{4\pi} - \bar{\kappa}_J J\right), \tag{2.8}$$

$$\frac{\partial H^i}{c\partial t} + \frac{\partial K^{ik}}{\partial x^k} = -\rho(\bar{\kappa} + \bar{\sigma})_H H^i, \tag{2.9}$$

where

$$j \equiv \int j_\nu d\nu, \tag{2.10}$$

$$\bar{\kappa}_I \equiv \frac{1}{I}\int (\kappa_\nu + \sigma_\nu)\, I_\nu d\nu, \tag{2.11}$$

$$\bar{\kappa}_J \equiv \frac{1}{J}\int \kappa_\nu J_\nu d\nu, \tag{2.12}$$

$$\bar{\sigma}_J \equiv \frac{1}{J}\int \sigma_\nu J_\nu d\nu, \tag{2.13}$$

$$(\bar{\kappa} + \bar{\sigma})_H \equiv \frac{1}{Hi}\int (\kappa_\nu + \sigma_\nu)\, H_\nu^i d\nu. \tag{2.14}$$

2.2.3 Closure Relation (Eddington Approximation)

$$K_\nu^{ij} = \frac{\delta^{ij}}{3} J_\nu. \tag{2.15}$$

2.2.4 Source Function

When the transfer equation (2.4) is expressed as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu = \rho\,(\kappa_\nu + \sigma_\nu)\,(S_\nu - I_\nu), \tag{2.16}$$

the *source function* S_ν is introduced as

$$S_\nu \equiv \frac{1}{\kappa_\nu + \sigma_\nu}\left(\frac{j_\nu}{4\pi} + \sigma_\nu J_\nu\right). \tag{2.17}$$

In the case of the local thermodynamic equilibrium (LTE), the source function (2.17) becomes

$$\begin{aligned} S_\nu &= \frac{1}{\kappa_\nu + \sigma_\nu}\left(\kappa_\nu B_\nu + \sigma_\nu J_\nu\right) \\ &= \varepsilon_\nu B_\nu + (1 - \varepsilon_\nu) J_\nu, \end{aligned} \tag{2.18}$$

where

$$\varepsilon_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} \tag{2.19}$$

is the *photon destruction probability* ($\varepsilon_\nu = 1 - A_\nu$).

In terms of this source function, for example, equation (2.4) and (2.5), and (2.6) with the Eddington approximation (2.15) are, respectively, reexpressed as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I_\nu = \rho\,(\kappa_\nu + \sigma_\nu)\,(S_\nu - I_\nu), \tag{2.20}$$

$$\frac{\partial J_\nu}{c\partial t} + \frac{\partial H_\nu^k}{\partial x^k} = \rho\,(\kappa_\nu + \sigma_\nu)\,(S_\nu - J_\nu), \tag{2.21}$$

$$\frac{\partial H_\nu^i}{c\partial t} + \frac{1}{3}\frac{\partial J_\nu}{\partial x^i} = -\rho(\kappa_\nu + \sigma_\nu)H_\nu^i. \tag{2.22}$$