第3章 Radiative Transfer Plane-Parallel Gray (Frequency-Integrated) variables $IEFP$ in physics

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3.1 Basic Equations

We here assume the followings: (i) The disk is steady and axisymmetric. (ii) It is also geometrically thin and plane parallel. (iii) As a closure relation, we use the Eddington approximation. (iv) The gray approximation, where the opacity does not depend on frequency, is adopted. (v) The viscous heating rate is concentrated at the equator or uniform in the vertical direction.

The radiative transfer equations are given in several literatures (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 1998). For the plane-parallel geometry in the vertical direction (z), the frequency-integrated transfer equation, the zeroth moment equation, and the first moment equation become, respectively,

$$\cos\theta\frac{dI}{dz} = \rho\left[\frac{j}{4\pi} - (\kappa_{\rm abs}+\kappa_{\rm sca})\,I + \kappa_{\rm sca}\frac{c}{4\pi}E\right], \tag{3.1}$$

$$\frac{dF}{dz} = \rho\,(j - c\kappa_{\rm abs}E), \tag{3.2}$$

$$\frac{dP}{dz} = -\frac{\rho(\kappa_{\rm abs}+\kappa_{\rm sca})}{c}F, \tag{3.3}$$

where θ is the polar angle, I the frequency-integrated specific intensity, E the radiation energy density, F the vertical component of the radiative flux, P the zz-component of the radiation stress tensor, ρ the gas density, and c the speed of light. The mass emissivity j and opacity $\kappa_{\rm abs}$ and $\kappa_{\rm sca}$ are assumed to be independent of the frequency (gray approximation).

For matter, the vertical momentum balance and energy equation are, respectively,

$$0 = -\frac{d\psi}{dz} - \frac{1}{\rho}\frac{dp}{dz} + \frac{\kappa_{\rm abs}+\kappa_{\rm sca}}{c}F, \tag{3.4}$$

$$0 = q^+_{\rm vis} - \rho\,(j - c\kappa_{\rm abs}E), \tag{3.5}$$

where ψ is the gravitational potential, p the gas pressure, and $q^+_{\rm vis}$ the viscous-heating rate. In this paper, we do not solve the hydrostatic equilibrium (4). Generally speaking, when the contribution of the radiative flux is small, compared with the pressure gradient term, the gas pressure dominates in the atmoshere, and the density distribution will not be constant. When the radiative flux is strong, on the other hand, the radiation pressure dominates, and the density may be approximately constant throughout much of the disk. Anyway, we suppose that the density distribution would be ajusted so as to hold the hydrostatic equilibrium (4) through the main part of the disk atmosphere, under the radiative flux obtained later.

Using this energy equation (3.5) and introducing the optical depth, defined by

$$d\tau \equiv -\rho\,(\kappa_{\rm abs}+\kappa_{\rm sca})\,dz, \tag{3.6}$$

we rewrite the radiative transfer equations:

$$\mu\frac{dI}{d\tau} = I - \frac{c}{4\pi}E - \frac{1}{4\pi}\frac{1}{\kappa_{\rm abs}+\kappa_{\rm sca}}\frac{q^+_{\rm vis}}{\rho}, \tag{3.7}$$

$$\frac{dF}{d\tau} = -\frac{1}{\kappa_{\rm abs}+\kappa_{\rm sca}}\frac{q^+_{\rm vis}}{\rho}, \tag{3.8}$$

$$c\frac{dP}{d\tau} = F, \tag{3.9}$$

$$cP = \frac{1}{3}cE, \tag{3.10}$$

where $\mu \equiv \cos\theta$. Final equation is the usual Eddington approximation.

As for the boundary condition at the disk surface of $\tau = 0$, we impose a usual condition:

$$3cP_\mathrm{s} = cE_\mathrm{s} = 2F_\mathrm{s} \qquad \text{at} \quad \tau = 0, \tag{3.11}$$

where the subscript s denotes the values at the disk surface.

For the internal heating, we consider two extreme cases: (i) No heating ($q^+_\mathrm{vis} = 0$), where the viscous heating is concentrated at the disk equator and there is no heating source in the atmosphere. (ii) Uniform heating in the sense that $q^+_\mathrm{vis}/(\kappa^\mathrm{abs} + \kappa^\mathrm{sca})\rho$ =constant. The latter case means that the kinematic viscosity ν is constant in the vertical direction, since $q^+_\mathrm{vis}/\rho = \nu(rd\Omega/dr)^2$, as long as the opacities are constant.

Finally, the disk total optical depth becomes

$$\tau_0 = -\int_H^0 \rho(\kappa_\mathrm{abs} + \kappa_\mathrm{sca})dz, \tag{3.12}$$

where H is the disk half-thickness.

3.2 Analytical Solutions

Except for the emergent intensity I, several analytical expressions for moments as well as temperature distributions were obtained by several researchers (e.g., Laor, Netzer 1989; Hubeny et al. 2005; Artemova et al. 1996). For the completeness, we recalculate them as well as the intensity I.

3.2.1 No Heating Case

We first consider the case without heating in the disk atmosphere: $q^+_\mathrm{vis} = 0$, but with uniform incident intensity I_0 from the disk equator, where the viscous heating is assumed to be concentrated.

In this case, the analytical solutions of moment equations are easily given as

$$F = F_\mathrm{s} = \pi I_0, \tag{3.13}$$

$$3cP = cE = 3F_\mathrm{s}\left(\frac{2}{3} + \tau\right). \tag{3.14}$$

This is a familar solution under the Milne-Eddington approximation for a plane-parallel geometry. It should be noted that the vertical radiative flux F is conserved, and equals to πI_0 at the disk equator.

Since we obtain the radiation energy density E in the explicit form, we can now integrate the radiative transfer equation (3.7). After several partial integrations, we obtain both an outward intensity $I(\tau, \mu)$ ($\mu > 0$) and an inward intensity $I(\tau, -\mu)$ as

$$I(\tau,\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3} + \tau + \mu - \left(\frac{2}{3} + \tau_0 + \mu\right)e^{(\tau-\tau_0)/\mu}\right] + I(\tau_0,\mu)e^{(\tau-\tau_0)/\mu} \tag{3.15}$$

$$I(\tau,-\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3} + \tau - \mu - \left(\frac{2}{3} - \mu\right)e^{-\tau/\mu}\right], \tag{3.16}$$

where $I(\tau_0, \mu)$ is the boundary value at the midplane of the disk.

In the geometrically thin disk with finite optical depth τ_0 and uniform incident intensity I_0 from the disk equator, the boundary value $I(\tau_0, \mu)$ of the outward intensity I consists of two parts:

$$I(\tau_0, \mu) = I_0 + I(\tau_0, -\mu), \tag{3.17}$$

where I_0 is the uniform incident intensity and $I(\tau_0, -\mu)$ is the *inward* intensity from the backside of the disk beyond the midplane. Determining $I(\tau_0, -\mu)$ from equation (3.16), we finally obtain the outward intensity as

$$\begin{aligned} I(\tau,\mu) &= \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3} + \tau + \mu - 2\mu e^{(\tau-\tau_0)/\mu} - \left(\frac{2}{3} - \mu\right)e^{(\tau-2\tau_0)/\mu}\right] + I_0 e^{(\tau-\tau_0)/\mu} \\ &= \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3} + \tau + \mu + \left(\frac{4}{3} - 2\mu\right)e^{(\tau-\tau_0)/\mu} - \left(\frac{2}{3} - \mu\right)e^{(\tau-2\tau_0)/\mu}\right] \end{aligned} \tag{3.18}$$

where we have used $F_\mathrm{s} = \pi I_0$.

For sufficiently large optical depth τ_0, this equation (3.18) reduces to the usual Milne-Eddington solution:

$$I = \frac{3F_\mathrm{s}}{4\pi}\left(\frac{2}{3} + \tau + \mu\right). \tag{3.19}$$

Finally, the emergent intensity $I(0, \mu)$ emitted from the disk surface for the finite optical depth becomes

$$I(0,\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3} + \mu + \left(\frac{4}{3} - 2\mu\right)e^{-\tau_0/\mu} - \left(\frac{2}{3} - \mu\right)e^{-2\tau_0/\mu}\right]. \tag{3.20}$$

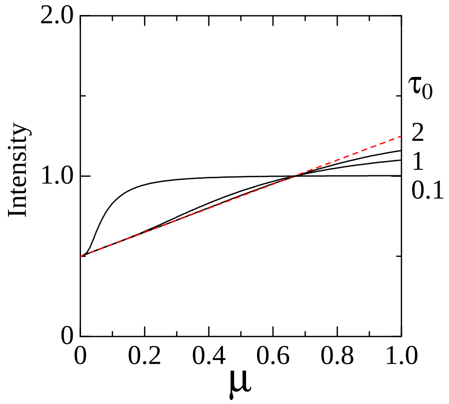

図 3.1: Normalized emergent intensity as a function of μ for the case without heating. The numbers attached on each curve are values of τ_0 at the disk midplane. The dashed straight line is for the usual plane-parallel case with infinite optical depth.

In figure 1, the emergent intensity $I(0,\mu)$ normalized by the isotropic value $\bar{I}$ $(= F_\mathrm{s}/\pi)$ is shown for several values of τ_0 as a function of μ.

As is easily seen in figure 1, for large optical depth ($\tau_0 > 10$) the angle-dependence of the emergent intensity is very close to the case for a usual plane-parallel case with infinite optical depth. Therefore, the usual limb-darkening effect is seen. Namely, in the case of a semi-infinite disk with large optical depth, the energy density increases linearly with the optical depth in the atmosphere, and the temperature increases accordingly. As a result, an observer at a pole-on position of $\mu = 1$ will see deeper in the disk, where the temperature (and therefore, the source function) is larger than that observed by an observer at an edge-on position of $\mu = 0$. Thus, the observed intensity will be higher at $\mu = 1$. This is just a usual limb-darkening.

For small optical depth, however, the angle-dependence is drastically changed. When the optical depth is a few, the vertical intensity ($\mu \sim 1$) decreases due to the finiteness of the optical depth. That is, we cannot see the 'deeper' position in the atmosphere, compared with the case of a semi-infinite disk. Furthermore, when the optical depth is less than unity, the intensity in the direction of small μ increases, and the emergent intensity becomes isotropic with a uniform value I_0 at the disk equator; the limb-darkening effect disappears. Indeed, the limiting case of $\tau_0 \sim 0$, $I(0,\mu) \sim F_\mathrm{s}/\pi$.

3.2.2 Uniform Heating Case

Now, we consider the case with uniform heating: $q^+_\mathrm{vis}/(\kappa_\mathrm{abs}+\kappa_\mathrm{sca})\rho =$constant. Integrating the equation (3.8) under the following boundary conditions:

$$F = 0 \quad \text{at} \quad \tau = \tau_0,$$
$$F = F_\mathrm{s} \quad \text{at} \quad \tau = 0, \tag{3.21}$$

we obtain

$$F = F_\mathrm{s}\left(1 - \frac{\tau}{\tau_0}\right). \tag{3.22}$$

The radiative flux F linearly increases from 0 to the surface value F_s.

Substituting equation (3.22) into equation (3.9), and integrating the resultant equation under the boundary condition (11), we obtain

$$3cP = cE = 3F_\mathrm{s}\left(\frac{2}{3} + \tau - \frac{\tau^2}{2\tau_0}\right). \tag{3.23}$$

This expression (3.23) for finite optical depth is seen in, e.g., Laor and Netzer (1989). A similar but more general expression was obtained by Hubeny (1990). In any case, this expression reduces to the Milne-Eddington solution for sufficiently large optical depth. In the case of finite optical depth, the radiation energy density and pressure decrease from the midplane to the surface in the quadratic form. It should be noted that at the midplane of the disk of $\tau = \tau_0$,

$$3cP = cE = 3F_\mathrm{s}\left(\frac{2}{3} + \frac{\tau_0}{2}\right). \tag{3.24}$$

As already mentioned by Hubeny (1990), the energy density at the disk midplane is the half of the corresponding stellar atmospheric one. This is explained by the fact that the radiation from the disk midplane may escape equally to both sides of the disk.

Since we obtain the radiation energy density E in the explicit form (3.23), we can now integrate the radiative transfer equation (3.7). After several partial integrations, we obtain both an outward intensity $I(\tau,\mu)$ ($\mu > 0$) and an inward intensity $I(\tau,-\mu)$ as

$$I(\tau,\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3}+\tau+\mu+\frac{1}{\tau_0}\left(\frac{1}{3}-\frac{\tau^2}{2}-\mu\tau-\mu^2\right)-\left(\frac{2}{3}+\frac{\tau_0}{2}+\frac{1}{3\tau_0}-\frac{\mu^2}{\tau_0}\right)e^{(\tau-\tau_0)/\mu}\right] + I(\tau_0,\mu)e^{(\tau-\tau_0)/\mu}, \tag{3.25}$$

$$I(\tau,-\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3}+\tau-\mu+\frac{1}{\tau_0}\left(\frac{1}{3}-\frac{\tau^2}{2}+\mu\tau-\mu^2\right)-\left(\frac{2}{3}-\mu+\frac{1}{3\tau_0}-\frac{\mu^2}{\tau_0}\right)e^{-\tau/\mu}\right] \tag{3.26}$$

where $I(\tau_0,\mu)$ is the boundary value at the midplane of the disk.

In the case with uniform heating and without the incident intensity, the boundary value $I(\tau_0,\mu)$ of the outward intensity I is

$$I(\tau_0,\mu) = I(\tau_0,-\mu), \tag{3.27}$$

and we finally obtain the outward intensity as

$$I(\tau,\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3}+\tau+\mu+\frac{1}{\tau_0}\left(\frac{1}{3}-\frac{\tau^2}{2}-\mu\tau-\mu^2\right)-\left(\frac{2}{3}-\mu+\frac{1}{3\tau_0}-\frac{\mu^2}{\tau_0}\right)e^{(\tau-2\tau_0)/\mu}\right] \tag{3.28}$$

(3.29)

For sufficiently large optical depth τ_0, this equation (3.28) also reduces to the usual Milne-Eddington solution (19).

Finally, the emergent intensity $I(0,\mu)$ emitted from the disk surface for the finite optical depth becomes

$$I(0,\mu) = \frac{3F_\mathrm{s}}{4\pi}\left[\frac{2}{3}+\mu+\frac{1}{\tau_0}\left(\frac{1}{3}-\mu^2\right)-\left(\frac{2}{3}-\mu+\frac{1}{3\tau_0}-\frac{\mu^2}{\tau_0}\right)e^{-2\tau_0/\mu}\right] \tag{3.30}$$

In figure 2, the emergent intensity $I(0,\mu)$ normalized by the isotropic value $\bar{I}$ ($= F_\mathrm{s}/\pi$) is shown for several values of τ_0 as a function of μ.

As is easily seen in figure 2, for large optical depth ($\tau_0 > 10$) the angle-dependence of the emergent intensity is very close to the case with a usual plane-parallel case with infinite optical depth. Therefore, the usual limb-darkening effect is seen, as already stated at the end of section 3.1.

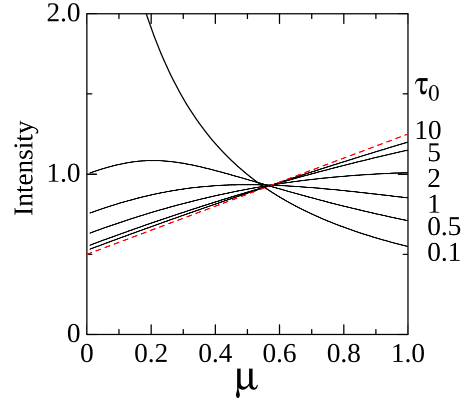

図 3.2: Normalized emergent intensity as a function of μ for the case with uniform heating. The numbers attached on each curve are values of τ_0 at the disk midplane. The dashed straight line is for the usual plane-parallel case with infinite optical depth.

For small optical depth, however, the angle-dependence is drastically changed similar to the case without heating. In the vertical direction of $\mu \sim 1$, the emergent intensity decreases as the optical depth decreases. This is due to the finiteness of the optical depth. That is, we cannot see the 'deeper' position in the atmosphere, compared with the case of a semi-infinite disk. In the inclined direction of small μ, on the other hand, the emergent intensity becomes larger than that in the case of the infinite optical depth. Moreover, when the optical depth is less than unity, the emergent intensity for small μ is greater than unity: the *limb brightening* takes place. Indeed, in the limiting case of $\tau_0 \sim 0$, $I(0,\mu) \sim (F_\mathrm{s}/\pi)/(2\mu)$. This is because that the path length is longer for such a case of small μ. That is, in this case for low optical depth, the source function is very uniform. This, coupled with the absence of an isotropic source at the midplane, is why the geometric effect (longer path length) is dominant and one finds limb 'brightening'.