第22章 Relativistic Radiative Transfer Plane-Parallel Comoving Frame variables $I_0 E_0 F_0 P_0$ in physics

After Fukue, J. 2011, PTP, 125, 837

22.1 Relativistic Radiative Transfer Equation

The radiative transfer equations are given in several literatures (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 1998, 2008; Mihalas, Auer 2001; Peraiah 2002; Castor 2004). The basic equations for relativistic radiation hydrodynamics are given in, e.g., the appendix E of Kato et al. (2008) in general and vertical forms (See also Fukue 2008c).

In a general form the radiative transfer equation in the mixed frame, where the variables in the inertial and comoving frames are used, is expressed as

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\boldsymbol{l}\cdot\boldsymbol{\nabla})\, I = \rho\gamma^3\left(1+\frac{\boldsymbol{v}\cdot\boldsymbol{l}_0}{c}\right)^3 \times \left[\frac{j_0}{4\pi} - \left(\kappa_0^{\rm abs}+\kappa_0^{\rm sca}\right) I_0 + \kappa_0^{\rm sca}\frac{cE_0}{4\pi}\right], \qquad (22.1)$$

Here, $\boldsymbol{v}$ is the flow velocity, c is the speed of light, and γ $(= 1/\sqrt{1-v^2/c^2})$ is the Lorentz factor. In the left-hand side the frequency-integrated specific intensity I and the direction cosine $\boldsymbol{l}$ are quantities measured in the inertial (fixed) frame. In the right-hand side, on the other hand, the mass density ρ, the frequency-integrated mass emissivity j_0, the frequency-integrated mass absorption coefficient $\kappa_0^{\rm abs}$, the frequency-integrated mass scattering coefficient $\kappa_0^{\rm sca}$, the frequency-integrated specific intensity I_0, the frequency-integrated radiation energy density E_0, and the direction cosine $\boldsymbol{l}_0$ are quantities measured in the comoving (fluid) frame. In this paper, instead of the weakly anisotropic Thomson scattering, we assume that the scattering is isotropic for simplicity.

Let us suppose a steady relativistic flow in the vertical direction. In the plane-parallel geometry with the vertical axis z the transfer equation is expressed as

$$\mu\frac{dI}{dz} = \rho\gamma^3(1+\beta\mu_0)^3\left[\frac{j_0}{4\pi} - \left(\kappa_0^{\rm abs}+\kappa_0^{\rm sca}\right) I_0 + \kappa_0^{\rm sca}\frac{cE_0}{4\pi}\right], \qquad (22.2)$$

where β $(= v/c)$ is the normalized vertical speed, and μ $(= \cos\theta)$ and μ_0 are the direction cosines in the inertial and comoving frames, respectively.

If the local thermodynamic equilibrium (LTE) holds in the comoving frame,

$$\frac{j_0}{4\pi} = \kappa_0^{\rm abs} B_0, \qquad (22.3)$$

where B_0 $(= \sigma T_0^4/\pi)$ is the frequency-integrated blackbody intensity in the comoving frame, T_0 being the blackbody temperature. Introducing the optical depth defined by

$$d\tau = -\left(\kappa_0^{\rm abs}+\kappa_0^{\rm sca}\right)\rho dz, \qquad (22.4)$$

the transfer equation (22.2) then becomes

$$\mu\frac{dI}{d\tau} = \gamma^3(1+\beta\mu_0)^3\left[I_0 - (1-A)B_0 - AJ_0\right], \qquad (22.5)$$

where

$$A \equiv \frac{\kappa_0^{\rm sca}}{\kappa_0^{\rm abs}+\kappa_0^{\rm sca}} \qquad (22.6)$$

is the scattering albedo and $J_0 = cE_0/(4\pi)$ is the mean intensity in the comoving frame.

The specific intensity I and the direction cosine μ in the inertial frame are related to the quantities in the comoving frame by

$$I(\tau,\mu) = \gamma^4(1+\beta\mu_0)^4 I_0(\tau,\mu_0), \qquad (22.7)$$

$$\mu = \frac{\mu_0 + \beta}{1 + \beta\mu_0}. \tag{22.8}$$

Inserting these relations to the transfer equation (22.5), we have the transfer equation in the comoving frame;

$$\gamma(\mu_0 + \beta)\frac{dI_0}{d\tau} + \gamma(\mu_0 + \beta) I_0 \frac{d}{d\tau} \ln\left[\gamma^4(1 + \beta\mu_0)^4\right]$$
$$= I_0 - AJ_0 - (1 - A)B_0. \tag{22.9}$$

It should be noted that the equivalent forms of this simple equation (22.9) are shown in several literatures (e.g., Mihalas, Mihalas 1984; Peraiah 2002; Castor 2004). However, no explicit analytical solutions are found, although in some lieterture (e.g., Peraiah 2002) the numerical solutions are shown and its meanings are discussed.

22.2 Linear-Flow Approximation

In this paper, the flow speed is assumed to be constant for simplicity, and we seek analytical solutions of the transfer equation (22.9) under appropriate situations and boundary conditions. We first examine the relativistic radiative transfer equation (22.9) under the linear-flow approximation in the comoving frame.

In the linear-flow approximation the comoving intensity $I_0(\tau, \mu_0)$ is serarated into two parts; the upward intensity $I_0^+(\tau)$ for $\mu_0 = +1$ and the downward intensity $I_0^-(\tau)$ for $\mu_0 = -1$. The transfer equation (22.9) is then separated into two equations:

$$\gamma(1 + \beta)\frac{dI_0^+}{d\tau} = I_0^+ - AJ_0 - (1 - A)B_0, \tag{22.10}$$

$$\gamma(1 - \beta)\frac{dI_0^-}{d\tau} = -I_0^- + AJ_0 + (1 - A)B_0. \tag{22.11}$$

In this case, the frequency-integrated mean intensity J_0 and the frequency-integrated net flux (Eddington flux) H_0 are expressed, respectively,

$$J_0 = \frac{cE_0}{4\pi} = \frac{1}{2}\left(I_0^+ + I_0^-\right), \tag{22.12}$$

$$H_0 = \frac{F_0}{4\pi} = \frac{1}{2}\left(I_0^+ - I_0^-\right), \tag{22.13}$$

where F_0 is the radiative flux in the comoving frame.

22.2.1 Pure Scattering Case

We first consider the pure scattering case of $A = 1$, or the radiative equilibrium (RE) case, where $j_0 = \kappa_0^{\rm abs} cE_0$. In this case the transfer equations (22.10) and (22.11) become

$$\gamma(1 + \beta)\frac{dI_0^+}{d\tau} = \frac{1}{2}I_0^+ - \frac{1}{2}I_0^-, \tag{22.14}$$

$$\gamma(1 - \beta)\frac{dI_0^-}{d\tau} = \frac{1}{2}I_0^+ - \frac{1}{2}I_0^-. \tag{22.15}$$

We easily integrate equations (22.14) and (22.15) to yield

$$(1 + \beta)I_0^+ - (1 - \beta)I_0^- = D_0 \text{ (const.)}, \tag{22.16}$$

and hence,

$$I_0^- = \frac{1 + \beta}{1 - \beta}I_0^+ - \frac{D_0}{1 - \beta}. \tag{22.17}$$

Inserting this equation (22.17) into equation (22.14), we obtain general solutions of these homogenious equations:

$$I_0^+ = C_0 e^{-\gamma\beta\tau} + \frac{D_0}{2\beta}, \tag{22.18}$$

$$I_0^- = C_0 \frac{1 + \beta}{1 - \beta} e^{-\gamma\beta\tau} + \frac{D_0}{2\beta}, \tag{22.19}$$

where C_0 is the integration costant.

We now impose the boundary condition at $\tau = 0$. We suppose that there is no irradiation at $\tau = 0$; $I_0^-(0) = 0$. Hence, the integration constant C_0 becomes

$$C_0 = -\frac{1 - \beta}{1 + \beta}\frac{D_0}{2\beta}. \tag{22.20}$$

Furthermore, we replace $D_0/2$ by H_{00}, which is the constant flux in the non-relativistic limit. Thus, we obtain the special solutions:

$$I_0^+ = \frac{H_{00}}{\beta}\left(1 - \frac{1-\beta}{1+\beta}e^{-\gamma\beta\tau}\right), \tag{22.21}$$

$$I_0^- = \frac{H_{00}}{\beta}\left(1 - e^{-\gamma\beta\tau}\right), \tag{22.22}$$

$$J_0 = \frac{H_{00}}{\beta}\left(1 - \frac{1}{1+\beta}e^{-\gamma\beta\tau}\right), \tag{22.23}$$

$$H_0 = \frac{H_{00}}{1+\beta}e^{-\gamma\beta\tau}. \tag{22.24}$$

In the non-relativistic limit of $\beta \sim 0$, these solutions reduce usual linear-flow solutions:

$$I_0^+ = H_{00}(\tau + 2), \tag{22.25}$$

$$I_0^- = H_{00}\tau, \tag{22.26}$$

$$J_0 = H_{00}(\tau + 1), \tag{22.27}$$

$$H_0 = H_{00}. \tag{22.28}$$

These analytical solutions in the comoving frame under the linear-flow approximation are shown in figure 1. In figure 1a the upward and downward intensities $I_0^\pm$ normalized by H_{00} are shown by solid and dashed curves, respectively, for various values of the flow velocity, whereas in figure 1b the mean intensity J_0 and the net flux H_0 normalized by H_{00} are shown by solid and dashed curves, respectively. The values of β are 0 to 0.9 in steps of 0.1.

Here, we emphasize two characteristic properties of the relativistic radiative transfer. The first is its *exponential nature* on the optical depth. Originally, the radiative intensity has exponential behavior, since it is the solution of the linear differential equation. However, in the analytical solutions like the linear-flow ones or the Milne-Eddington ones in the non-relativistic radiative transfer problems the quantities are proportional to the optical depth. In the relativistic radiative transfer such exponential nature revives. As a result, the intensities approach constant values, while the net flux becomes zero, as an optical depth increases. This exponential properties of the radiative quantites

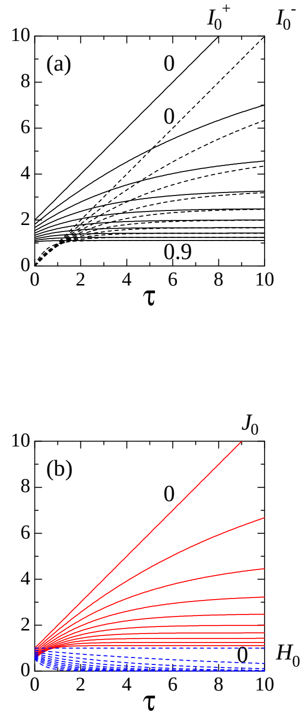

図 22.1: Linear-flow solutions in the comoving frame of relativistic plane-parallel flows for various flow speeds: (a) Normalized upward and downward intensities $I_0^\pm/H_{00}$ (solid and dashed curves, respectively), (b) Normalized mean intensity J_0/H_{00} and normalized net flux H_0/H_{00} (solid and dashed curves, respectively). The values of β are 0 to 0.9 in steps of 0.1 from top to bottom for each curve.

in the comoving frame were firstly found in Fukue (2008c) for the relativistic Milne-Eddington solution. In the present linear-flow treatment the exponential nature is clearly derived and shown.

It should be noted that Peraiah (1987) numerically solved the relativistic transfer equation in the plane-parallel case, and obtained, e.g., the mean intensity in the comoving frame, although he did not show the analytical form. These properties originate from the aberration effect, as Peraiah (1987) already mentioned.

The second is the relativistic modification on the *apparent/effective optical depth*. In the relativistic radiative transfer the optical depth τ in the exponential term is changed to be $\Gamma\tau$, where Γ is generally a function of the velocity β and its gradient. For example, in the present linear-flow solutions the effective optical depth becomes $\gamma\beta\tau$ as is seen in the solutions. This change of the apparent/effective optical depth originates from the relativistic Lorentz contraction (see, e.g., Abramowicz et al. 1991; Sumitomo et al. 2007; Fukue, Sumitomo 2009), and also found in, e.g., Fukue (2008c). However, in the present linear-flow treatment the change of the effective optical depth is also clearly shown analytically.

As for the quantities in the inertial frame, we transform the comoving quantities to the inertial ones by the Lorentz transformation. For example, the specific intensity in the inertial frame is expressed by equation (22.7). Hence, in the present linear-flow approximation the upward intensity $I^+(\tau)$ and the downward intensity $I^-(\tau)$ in the inertial frame are given respectively by

$$I^+ = I_0^+\gamma^4(1+\beta)^4 = I_0^+\left(\frac{1+\beta}{1-\beta}\right)^2, \quad (22.29)$$

$$I^- = I_0^+\gamma^4(1-\beta)^4 = I_0^+\left(\frac{1-\beta}{1+\beta}\right)^2. \quad (22.30)$$

Because of the aberration between the inertial and comoving frames, equation (22.8), the mean intensity J and net flux H in the inertial frame are respectively expressed as

$$J = \frac{1}{2}\left[\int_\beta^1 I^+ d\mu + \int_{-1}^\beta I^- d\mu\right]$$

$$= \frac{1}{2}\left[(1-\beta)I^+ + (1+\beta)I^-\right], \quad (22.31)$$

$$H = \frac{1}{2}\left[\int_\beta^1 I^+ \mu d\mu + \int_{-1}^\beta I^- \mu d\mu\right]$$

$$= \frac{1}{2}\left[(1-\beta^2)I^+ - (1+\beta^2)I^-\right]. \quad (22.32)$$

These analytical solutions in the inertial frame are shown in figure 2. In figure 2a the upward and downward intensities $I^\pm$ normalized by H_{00} are shown by thick solid and dashed curves, respectively, for various values of the flow velocity, whereas in figure 2b the mean intensity J and the net flux H normalized by H_{00} are shown by thick solid and dashed curves, respectively. The values of β are 0, 0.1, 0.2, 0.3, 0.4, 0.5.

As is seen in figure 2a, the upward intensity in the inertial frame is Doppler-boosted, while the downward intensity is Doppler-de-boosted, as the flow speed increases. The mean intensity and the net flux are also changed by the Doppler effect. In addition, when the flow speed is finite, the radiative quantities in the inertial frame approach constant values, as the optical depth increases.

These properties of radiative quantities in the comoving and inertial frames are qualitatively similar to those of the relativistic Milne-Eddington solution found by Fukue (2008c). In Fukue (2008c), however, due to the assumption on the Eddington factor, the flow speed is restricted in the range of $\beta < 1/\sqrt{3}$, whereas there is no limitation on the flow speed in the present linear-flow approximation.

22.2.2 General Case

We next consider the general case of $A \neq 1$. In this case the transfer equations (22.10) and (22.11) are explicitly written as

$$\gamma(1+\beta)\frac{dI_0^+}{d\tau} = \left(1-\frac{A}{2}\right)I_0^+ - \frac{A}{2}I_0^- - (1-A)B_0, \quad (22.33)$$

$$\gamma(1-\beta)\frac{dI_0^-}{d\tau} = -\left(1-\frac{A}{2}\right)I_0^- + \frac{A}{2}I_0^+ + (1-A)B_0. \quad (22.34)$$

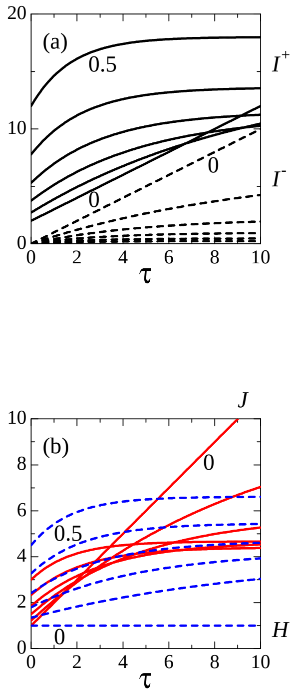

図 22.2: Linear-flow solutions in the inertial frame of relativistic plane-parallel flows for various flow speeds: (a) Normalized upward and downward intensities $I^{\pm}/H_{00}$ (thick solid and dashed curves, respectively), (b) Normalized mean intensity J/H_{00} and normalized net flux H/H_{00} (thick solid and dashed curves, respectively). The values of β are 0 to 0.5 in steps of 0.1 from bottom to top for I^{+} and H and from top to bottom for I^{-} and J.

The general solutions of the homogenious part of these linear equations (22.33) and (22.34) are

$$I_0^+ = C_1 e^{\Gamma\tau} + C_2 e^{-\Gamma\tau}, \tag{22.35}$$

$$I_0^- = C_3 e^{\Gamma\tau} + C_4 e^{-\Gamma\tau}, \tag{22.36}$$

where the coefficients C_i's and index Γ are functions of β in the present relativistic case.

Inserting these solutions into equations (22.33) and (22.34), we can obtain the form of Γ, and two relations among the coefficients. As a result, the general solutions (22.35) and (22.36) are expressed as

$$I_0^+ = C_1 e^{\Gamma\tau} + C_2 e^{-\Gamma\tau}, \tag{22.37}$$

$$I_0^- = C_1 P e^{\Gamma\tau} + C_2 Q e^{-\Gamma\tau}, \tag{22.38}$$

where

$$\Gamma = -\left(1-\frac{A}{2}\right)\gamma\beta \pm \sqrt{\left(1-\frac{A}{2}\right)^2\gamma^2\beta^2 + 1 - A}, \tag{22.39}$$

$$P = \frac{2-A-2\gamma(1+\beta)\Gamma}{A} = \frac{A}{2-A+2\gamma(1-\beta)\Gamma}, \tag{22.40}$$

$$Q = \frac{2-A+2\gamma(1+\beta)\Gamma}{A} = \frac{A}{2-a-2\gamma(1-\beta)\Gamma}. \tag{22.41}$$

In the limit of $A=1$, these solutions reduce to those of the pure scattering case.

As is well-known, the solutions of the inhomogenious equation are the combination of the general solutions of the homogenious part and the special solution of the inhomogenious part. When B_0 is constant, after some manipulations, we finally obtain the solutions for equations (22.33) and (22.34) as

$$I_0^+ = C_1 e^{\Gamma\tau} + C_2 e^{-\Gamma\tau} + \frac{1-\beta+\beta A}{(1+\beta)\Gamma^2}(1-A)B_0, \tag{22.42}$$

$$I_0^- = C_1 P e^{\Gamma\tau} + C_2 Q e^{-\Gamma\tau} - \frac{(1+\beta)(P+Q)+2(1-\beta)PQ}{(P-Q)\Gamma}\gamma(1-A)B_0. \tag{22.43}$$

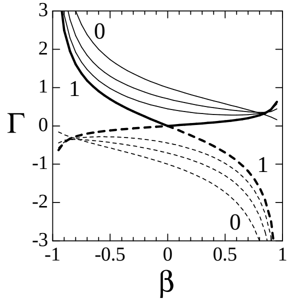

図 22.3: Index Γ as a function of β for several values of A $(= 0, 0.5, 0.8, 1)$. Solid and dashed curves correspond to the cases of plus and minus signs, respectively.

It should be noted that in these solutions the coefficients C_1 and C_2 are determined by the boundary conditions.

Since solutions of the general case have somewhat complicated forms, we do not show them, but show the exponential index Γ in figure 3 as a function of β for several values of A $(= 0, 0.5, 0.8, 1)$. Solid and dashed curves correspond to the cases of plus and minus signs, respectively. As the flow speed β increases, the index Γ increases and the effective optical depth $\Gamma\tau$ increases. This is the relativistic effect (cf. Abramowicz et al. 1991). In addition, As the albedo A decreases, the index Γ increases. This behavior exists in the non-relativistic case.

22.3 Two-Stream Approximation

In this section we examine the relativistic radiative transfer equation (22.9) under the two-stream approximation in the comoving frame.

In the two-stream approximation the comoving intensity $I_0(\tau, \mu_0)$ is serarated into two parts; the averaged upward intensity $I_0^+(\tau)$ to some typical direction $\bar{\mu}_0^+$ and the averaged downward intensity $I_0^-(\tau)$ to some typical direction $\bar{\mu}_0^-$.

The transfer equation (22.9) is also separated into two equations (cf. Thomas, Stamnes 1999):

$$\gamma\left(\bar{\mu}_0^+ + \beta\right)\frac{dI_0^+}{d\tau} = I_0^+ - \frac{A}{2}I_0^+ - \frac{A}{2}I_0^- - (1-A)B_0, \qquad (22.44)$$

$$\gamma\left(\bar{\mu}_0^- - \beta\right)\frac{dI_0^-}{d\tau} = -I_0^- + \frac{A}{2}I_0^+ + \frac{A}{2}I_0^- + (1-A)B_0, \qquad (22.45)$$

where

$$\bar{\mu}_0^\pm \equiv \frac{\int_0^1 \mu_0 I_0^\pm(\tau,\mu_0)d\mu_0}{\int_0^1 I_0^\pm(\tau,\mu_0)d\mu_0}. \qquad (22.46)$$

In this case, the frequency-integrated mean intensity J_0, the frequency-integrated net flux (Eddington flux) H_0, the frequency-integrated mean radiation pressure (K-integral), are expressed, respectively,

$$J_0 = \frac{cE_0}{4\pi} = \frac{1}{2}\left(I_0^+ + I_0^-\right), \qquad (22.47)$$

$$H_0 = \frac{F_0}{4\pi} \sim \frac{1}{2}\left(\bar{\mu}_0^+ I_0^+ - \bar{\mu}_0^- I_0^-\right), \qquad (22.48)$$

$$K_0 = \frac{cP_0}{4\pi} \sim \frac{1}{2}\left(\bar{\mu}_0^{2+} I_0^+ + \bar{\mu}_0^{2-} I_0^-\right). \qquad (22.49)$$

In the present study we assume that $\bar{\mu}_0^+ = \bar{\mu}_0^- = \bar{\mu}_0$ and $\bar{\mu}_0^{2+} = \bar{\mu}_0^{2-} = (\bar{\mu}_0)^2$.

22.3.1 Pure Scattering Case

We first consider the pure scattering case of $A = 1$, or the radiative equilibrium (RE) case, where $j_0 = \kappa_0^{\rm abs} cE_0$. In this case the transfer equations (22.44) and (22.45) become

$$\gamma\left(\bar{\mu}_0^+ + \beta\right)\frac{dI_0^+}{d\tau} = \frac{1}{2}I_0^+ - \frac{1}{2}I_0^-, \qquad (22.50)$$

$$\gamma\left(\bar{\mu}_0^- - \beta\right)\frac{dI_0^-}{d\tau} = \frac{1}{2}I_0^+ - \frac{1}{2}I_0^-. \qquad (22.51)$$

Using the similar procedure in the linear-flow case and the boundary condition at $\tau = 0$, we have analytical solutions for intensities,

$$I_0^+ = \frac{H_{00}}{\beta}\left(1 - \frac{\bar{\mu}_0 - \beta}{\bar{\mu}_0 + \beta}e^{-\Gamma\tau}\right), \qquad (22.52)$$

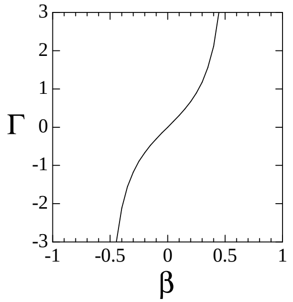

図 22.4: Index Γ as a function of β. The direction cosine is fixed as $\bar{\mu}_0^2 = f = 1/3$.

$$I_0^- = \frac{H_{00}}{\beta}\left(1 - e^{-\Gamma\tau}\right), \tag{22.53}$$

where

$$\Gamma = \frac{\beta}{\gamma(\bar{\mu}_0^2 - \beta^2)}. \tag{22.54}$$

Radiative moments become

$$J_0 = \frac{H_{00}}{\beta}\left(1 - \frac{\bar{\mu}_0}{\bar{\mu}_0 + \beta}e^{-\Gamma\tau}\right), \tag{22.55}$$

$$H_0 = \frac{H_{00}\bar{\mu}_0}{\bar{\mu}_0 + \beta}e^{-\Gamma\tau}, \tag{22.56}$$

$$K_0 = \frac{H_{00}\bar{\mu}_0^2}{\beta}\left(1 - \frac{\bar{\mu}_0}{\bar{\mu}_0 + \beta}e^{-\Gamma\tau}\right), \tag{22.57}$$

and therefore, the Eddington factor is

$$f \equiv \frac{K_0}{J_0} = \bar{\mu}_0^2. \tag{22.58}$$

In figure 4 the index Γ is shown as a function of β in the case of $\bar{\mu}_0^2 = f = 1/3$. As is seen in figure 4, As the flow speed β increases, the index Γ increases and the effective optical depth $\Gamma\tau$ also increases.

In the non-relativistic limit of $\beta \sim 0$, these solutions reduce usual two-stream solutions (see, e.g., Thomas, Stamnes 1999). If we set $\bar{\mu}_0 = 1$, on the other hand, these solutions reduce to those of the linear-flow approximation. In addition, if we set $\bar{\mu}_0^2 = f$, these solutions are quite similar to those of the Milne-Eddington case (Fukue 2008c).

Here, we mention the critical properties of the index Γ. As is seen in equation (22.54) and in figure 4, the index Γ diverges at $\beta = \pm\mu_0$ ($= \pm 1/\sqrt{3}$ in this case). This violation is due to the effect of *aberration*. Namely, the direction $\mu_0 = -\beta$ in the comoving frame corresponds to the direction $\mu = 0$ in the inertial frame. Hence, the two-stream approximation violates in the inertial frame at $\mu_0 = -\beta$. As a result, if we set $f = \bar{\mu}_0^2 = 1/3$ like a usual Eddington factor, then the present two-stream approximation is limited in the range of $\beta^2 < 1/3$ (cf. the Milne-Eddington case in Fukue 2008c). This sigular behavior will be discussed again in the last section.

On the contrary to the non-relativistic case, where the Eddington factor is 1/3 and some typical direction μ_0 is also constant ($\sim 1/2 - 1/\sqrt{2}$), there are no assurance that neither the Eddington factor f nor the typical direction μ_0 is constant in the relativistic case. Indeed, under the local approximation the relativistic Eddington factor does generally depend both on the velocity and its gradient (e.g., Fukue 2008b, 2008d, 2009a). Hence, in the present case the typical direction μ_0 in the comoving frame would generally depend on the velocity and its gradient, as a manner that $\mu_0 \to 1$ as $\beta \to 1$. However, at the present stage we have little knowledge on the precise solutions of the relativistic radiative transfer in such an extremely relativistic limit. Thus, the determination of the form of μ_0 is a future work; see, however, Fukue (2009a), where $f = (1 + 3\beta^2)/(3 + \beta^2)$ was found to be a good approximation in some cases.

22.3.2 General Case

We next consider the general case of $A \neq 1$ briefly. In this case the transfer equations (22.44) and (22.45) are explicitly written as

$$\gamma\left(\bar{\mu}_0^+ + \beta\right)\frac{dI_0^+}{d\tau} = \left(1-\frac{A}{2}\right)I_0^+ - \frac{A}{2}I_0^- - (1-A)B_0, \tag{22.59}$$

$$\gamma\left(\bar{\mu}_0^- - \beta\right)\frac{dI_0^-}{d\tau} = -\left(1-\frac{A}{2}\right)I_0^- + \frac{A}{2}I_0^+ + (1-A)B_0. \tag{22.60}$$

Procedures and solutions are similar to those in the linear-flow case, but the solutions have somewhat complicated forms, and therefore, we only show the exponential index,

$$\Gamma = -\frac{\left(1-\frac{A}{2}\right)\gamma\beta}{\gamma^2(\bar{\mu}_0^2-\beta^2)} \pm \frac{\sqrt{\left(1-\frac{A}{2}\right)^2\gamma^2\beta^2+(1-A)\gamma^2(\bar{\mu}_0^2-\beta^2)}}{\gamma^2(\bar{\mu}_0^2-\beta^2)}. \tag{22.61}$$

In the non-relativistic limit of $\beta \sim 0$, this index tends to $\Gamma \sim \sqrt{1-A}/\bar{\mu}_0$ (Thomas, Stamnes 1999).

In figure 5 the index Γ is shown as a function of β for several values of A $(=0, 0.5, 0.8, 1)$ in the case of $\bar{\mu}_0^2 = f = 1/3$. Solid and dashed curves correspond to the cases of plus and minus signs, respectively. As the flow speed increases and/or the albedo decreases, the index Γ increases and the effective optical depth increases.

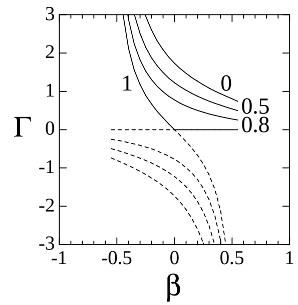

図 22.5: Index Γ as a function of β for several values of A $(=0, 0.5, 0.8, 1)$ in the case of $\bar{\mu}_0^2 = f = 1/3$. Solid and dashed curves correspond to the cases of plus and minus signs, respectively.